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# A NORMAL FORM IN THE HOMEOTOPY GROUP OF A SURFACE OF GENUS 2, WITH APPLICATIONS TO 3-MANIFOLDS

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ABSTRACT. It is shown that elements in the homeotopy group of a closed, compact, orientable 2-manifold of genus 2 can be put into a unique normal form which allows them to be enumerated systematically. As an application, the class of 3-manifolds which admit Heegaard splittings of genus 2 are shown to be denumerable, and a procedure is given for enumerating presentations for their fundamental groups.

1. **Introduction.** Let $T$ be a closed, compact orientable 2-manifold of genus 2, and let $H$ be its homeotopy group, that is, the group of all homeomorphisms of $T\to T$ modulo the subgroup of those homeomorphisms which are isotopic to the identity. Generators and defining relations for the group $H$ (or rather a subgroup of index 2 in $H$) were determined in [**1**]. We will show here that the elements in $H$ can be put into a unique normal form which shows that they are denumerable, and we will give an algorithm for enumerating them systematically.

This result has potential applications in the study of 3-manifolds. Define a "genus 2 Heegaard splitting" to be a compact 3-manifold $M$ without boundary which admits a decomposition as $M=X\cup X'$, where $X$ and $X'$ are solid handle-bodies of genus 2, and $X\cap X'=\text{bd}\, X=\text{bd}\, X'$. Each such splitting can be associated (nonuniquely) with an element in $H$, so that our enumeration of the elements in $H$ is equivalent to an enumeration (with duplications) of all possible genus 2 Heegaard splittings. We show how to use our enumeration to enumerate presentations (in terms of 2-generators, 2 relators) of the fundamental groups of all genus 2 Heegaard splittings. This enumeration might be useful as a tool in studying the classification of 3-manifolds.

At the conclusion we discuss briefly why the approach used here does not generalize to genus $>2$.

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2. **Enumerating the elements in** $H$. Our enumeration will be based on the following reasoning: we will first show that the structure of $H$ is closely related to the structure of the 5-string Artin braid group, $B_5$. We will then use results due to F. Garside [3] on the braid group to develop a normal form in $H$.

Instead of $H$, we first consider the mapping class group $M$ of the surface $T$, where $M$ is defined to be the subgroup of $H$ in which admissible maps are restricted to those homeomorphisms of $T \to T$ which preserve orientation. Using the well-known fact that the group $H$ is isomorphic to Aut $\pi_1 T$/Inn $\pi_1 T$, we can interpret elements of $H$ and $M$ as classes of automorphisms of the fundamental group $\pi_1 T$ of the surface $T$. Let $\pi_1 T$ be presented in terms of canonical generators $a_1$, $b_1$, $a_2$, $b_2$ which satisfy the single relation $[a_1, b_1][a_2, b_2]=1$. Let $\tau_1, \cdots, \tau_5$ be the automorphism classes of:

$$
\begin{aligned}
&\tau_1 : a_1 \to a_1 b_1^{-1},\\
&\tau_2 : b_1 \to b_1 a_1,\\
&\tau_3 : a_1 \to a_1 b_1^{-1} a_2 b_2 a_2^{-1},\\
&\quad\; a_2 \to a_2 b_2^{-1} a_2^{-1} b_1 a_2,\\
&\quad\; b_1 \to a_2 b_2^{-1} a_2^{-1} b_1 a_2 b_2 a_2^{-1},\\
&\tau_4 : b_2 \to b_2 a_2,\\
&\tau_5 : a_2 \to a_2 b_2^{-1},
\end{aligned}
\tag{1}
$$

where all generators which are not listed explicitly are assumed to be kept fixed. Then $M$ admits the presentation [**1**]:

*generators*: $\tau_1, \cdots, \tau_5$,

*relations*:

$$[\tau_i, \tau_j] = 1 \quad \text{if } |i-j| \geqq 2,\ 1 \leqq i, j \leqq 5, \tag{2}$$

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \qquad 1 \leqq i \leqq 4, \tag{3}$$

$$(\tau_1 \tau_2 \tau_3 \tau_4 \tau_5)^6 = 1, \tag{4}$$

$$(\tau_1 \tau_2 \tau_3 \tau_4 \tau_5^2 \tau_4 \tau_3 \tau_2 \tau_1)^2 = 1, \tag{5}$$

$$[\tau_1 \tau_2 \tau_3 \tau_4 \tau_5^2 \tau_4 \tau_3 \tau_2 \tau_1, \tau_i] = 1, \qquad 1 \leqq i \leqq 5. \tag{6}$$

The group $H$ is an extension of $M$ by a cyclic group or order 2. Define a new element $\tau_6 \in H$ by the rule:

$$
\begin{aligned}
\tau_6 : a_1 &\to b_1,\\
a_2 &\to b_2,\\
b_1 &\to a_1,\\
b_2 &\to a_2.
\end{aligned}
\tag{7}
$$

Since $\tau_6$ is orientation-reversing, the pair $(1, \tau_6)$ can be used as coset representatives for $M$ in $H$. Thus if we can show that the elements in $M$ are denumerable, it will follow that those in $H$ are also denumerable, with each element $x \in M$ corresponding to two elements $x$ and $x\tau_6$ in $H$.

To develop a normal form for elements in $M$, we first observe that it follows from relations (2)–(6) that the element $\alpha = \tau_1\tau_2\tau_3\tau_4\tau_5^2\tau_4\tau_3\tau_2\tau_1$ is in the center of $M$ and has order 2. Let $G$ be the quotient group obtained by setting $\alpha = 1$, and let $\rho_i \in G$ be the image of $\tau_i \in M$ under the homomorphism from $M \to G$. Then $G$ is generated by $\rho_1, \cdots, \rho_5$ and has defining relations $(2)_\rho$, $(3)_\rho$, $(4)_\rho$ and:

$$\rho_1\rho_2\rho_3\rho_4\rho_5^2\rho_4\rho_3\rho_2\rho_1 = 1 \tag{8}$$

where $(2)_\rho$, $(3)_\rho$, $(4)_\rho$ mean relations (2), (3), (4) but with the $\rho$'s substituted for the $\tau$'s. If we could list elements in $G$, we could list elements in $M$, by lifting the element $\rho_{\mu_1}^{\varepsilon_1} \cdots \rho_{\mu_r}^{\varepsilon_r}$ in $G$ to the pair $\tau_{\mu_1}^{\varepsilon_1} \cdots \tau_{\mu_r}^{\varepsilon_r}$ and $\alpha\tau_{\mu_1}^{\varepsilon_1} \cdots \tau_{\mu_r}^{\varepsilon_1}$ in $M$.

Now we note that if we add to relations $(2)_\rho$, $(3)_\rho$, $(4)_\rho$ and (8) the additional relation that the square of every generator is trivial, then our presentation for $G$ will go over to a presentation for the symmetric group $\Sigma_6$ on 6 letters [**2**, equation 6.28], each $\rho_i$ being mapped onto the transposition $(i, i+1)$. Let $K$ be the subgroup of $G$ consisting of all elements in $G$ whose images in $\Sigma_6$ leave the letter 6 invariant. Then $K$ has index 6 in $G$, and each element $\beta \in G$ has a unique representation $\beta = \gamma k_\mu$, where $\gamma \in K$ and the elements $k_0, \cdots, k_6$ are coset representatives for $K$ in $G$, with

$$k_0 = 1, \qquad k_\mu = \rho_5\rho_4 \cdots \rho_\mu \quad \text{if } \mu = 1, \cdots, 5.$$

Thus our enumeration is reduced to an enumeration in $K$. Since the elements $k_0, k_1, \cdots, k_5$ are a Schreier set of coset representatives for $K$ in $G$, we can apply the Schreier-Reidemeister method [**4**] to the known presentation for $G$ to obtain a presentation for $K$. This calculation gives that $K$ is generated by $\rho_1$, $\rho_2$, $\rho_3$, $\rho_4$ and has defining relations:

$$\rho_i\rho_j = \rho_j\rho_i, \qquad 1 \leqq i, j \leqq 4, \ |i-j| \geqq 2, \tag{9}$$

$$\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}, \qquad 1 \leqq i \leqq 4, \tag{10}$$

$$(\rho_1\rho_2\rho_3\rho_4)^5 = 1. \tag{11}$$

To understand the group $K$, we compare it to a group which is closely related to it, the 5-string Artin braid group, $B_5$. The $n$-string braid group $B_n$ can be defined as the abstract group with generators $\sigma_1, \cdots, \sigma_{n-1}$ and defining relations $(9)_\sigma$ and $(10)_\sigma$ where the indices $i, j$ are allowed to run between 1 and $n-1$ instead of between 1 and 4. Now, the element

$(\sigma_1\sigma_2 \cdots \sigma_{n-1})^n$ generates the center $Z(B_n)$ of $B_n$ (see, for example, [3]). Hence every element $\delta \in B_n$ can be written uniquely in the form

$$\delta = (\sigma_1\sigma_2 \cdots \sigma_{n-1})^{nm} V(\sigma_1, \cdots, \sigma_{n-1})$$

where $m$ is an integer. Under the natural homomorphism from $B_5 \to K$ the element $(\sigma_1\sigma_2\sigma_3\sigma_4)^{5m} V(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ in $B_5$ goes over to the element $V(\rho_1, \rho_2, \rho_3, \rho_4)$ in $K$, where two words $V$ and $V'$ in $K$ will be equivalent iff their preimages in $B_5$ are equivalent for some fixed $m$. Thus if we can develop a scheme for enumerating the distinct coset representatives of the center $Z(B_5)$ in $B_5$, we will also be able to enumerate the distinct elements of $K$, which will in turn enable us to enumerate elements in $G$, in $M$, and finally in $H$.

The braid group $B_n$ was studied by Garside in [3]. Garside developed a normal form for words in $B_n$, which we claim allows us to enumerate the coset representatives of $Z(B_n)$. In $B_n$, define the elements:

$$(12) \quad \Delta = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2)(\sigma_1), \qquad \phi = (\sigma_1 \cdots \sigma_{n-1})^n.$$

The element $\phi$ generates $Z(B_n)$. Moreover, as a consequence of relations $(9)_\sigma$ and $(10)_\sigma$ one can show that in $B_n$:

$$(13) \qquad \Delta^2 = \phi.$$

Now, Garside finds coset representatives for the infinite cyclic subgroup of $B_n$ generated by $\Delta$. If we denote his coset representatives by $\{P_i, i \in I\}$, then it is immediate that $\{P_i, \Delta P_i; i \in I\}$ will be a set of coset representatives for $Z(B_n)$. So we need only show that Garside's coset representatives are denumerable.

The words $P_i$ obtained by Garside have a very special form which we describe briefly. Let $S_n$ be the semigroup with generators $\sigma_1, \cdots, \sigma_{n-1}$ and with the same defining relations as $B_n$. Garside shows that $S_n$ can be imbedded in $B_n$, and that the coset representatives $\{P_i, i \in I\}$ can be chosen to be in $S_n$, i.e. so that they involve only positive powers of the $\sigma_i$. He also gives a procedure for testing whether a given $P(\sigma_1, \cdots, \sigma_{n-1}) \in S_n$ is a coset representative (i.e. is "prime to $\Delta$" and is "lexicographically smallest" in the "diagram of $P$" in Garside's notation; for details see [3]). Garside's procedure for finding the coset representative of a given word goes over immediately to a procedure for listing coset representatives: one simply lists all words in $S_n$ in order of increasing length, finds the coset representative of each such word, and enters it in the list if it is not a coset representative which has appeared earlier on the list.

It then follows that the elements of our original group $H$ are also denumerable, because each coset representative $P(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ of the cyclic subgroup $\langle\Delta\rangle$ in $B_5$ corresponds to precisely 48 distinct words $W(\tau_1, \cdots, \tau_6) \in H$, i.e., the words:

$$W(\tau_1, \cdots, \tau_6) = \alpha^{\varepsilon_1}\Delta^{\varepsilon_2}P(\tau_1, \tau_2, \tau_3, \tau_4)k_\mu\tau_6^{\varepsilon_3}$$

where:

$$\alpha = \tau_1\tau_2\tau_3\tau_4\tau_5^2\tau_4\tau_3\tau_2\tau_1, \qquad \Delta = \tau_1\tau_2\tau_3\tau_4\tau_1\tau_2\tau_3\tau_1\tau_2\tau_1,$$

$$\mu = 0, 1, 2, 3, 4, 5,$$

$$\begin{aligned} k_\mu &= \tau_5\tau_4\cdots\tau_\mu \quad && \text{if } \mu = 1, 2, 3, 4, 5, \\ &= 1 && \text{if } \mu = 0, \end{aligned}$$

$$\varepsilon_1, \varepsilon_2, \varepsilon_3 = 0 \quad \text{or } 1.$$

3. **Enumerating the fundamental groups of the genus 2 Heegaard splittings.** Let $\tau \in H$, and suppose that $\tau$ is represented by an automorphism which acts on $\pi_1 T$ as follows:

$$\begin{aligned} \tau: a_i &\to A_i(a_1, a_2, b_1, b_2), \\ b_i &\to B_i(a_1, a_2, b_1, b_2), \qquad i = 1, 2. \end{aligned}$$

Then it is shown in [**5**, Theorem 10.1] that a genus 2 Heegaard splitting exists with fundamental group:

$$\pi_1(X \cup X') = \langle a_1, a_2; A_i(a_1, a_2, 1, 1), i = 1, 2\rangle. \tag{14}$$

The enumeration procedure described in §2 can be used to obtain an enumeration of the relators $A_i(a_1, a_2, 1, 1)$, $i=1, 2$, by using equation (1) and (7) to compute the action of $\tau$ on the elements $a_1, a_2 \in \pi_1 T$ for each $\tau \in H$. Thus we could enumerate the fundamental groups of all 3-manifolds which admit genus 2 Heegaard splittings. Of course there will be duplications, both because the same words $A_1$ and $A_2$ will appear repeatedly, and more seriously because no procedure exists for distinguishing isomorphism classes of groups which are defined by presentations.

4. **Generalizing to genus** $>2$. The analysis given above does not carry over to genus $>2$, because it depends crucially on the presentation obtained in [**1**] for $M$, which generalizes to a presentation for a *proper subgroup* of the mapping class group $M(T_g)$ of a closed, compact orientable surface $T_g$ of genus $g>2$. The algebraic relationship between this subgroup and the full group $M(T_g)$ is not, to the author's knowledge, well understood.

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