REPRESENTATION OF THE PROJECTABLE AND STRONGLY
PROJECTABLE HULLS OF A LATTICE-ORDERED GROUP

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ABSTRACT. A representable $l$-group $G$ can be embedded into a
projectable (strongly projectable) $l$-group; an essential extension
of $G$ which is minimal with respect to being projectable (strongly
projectable) is unique. In this paper these projectable and strongly
projectable “hulls” of $G$ are represented using direct limits. If $G$
is an $f$-ring, or $f$-ring without nonzero nilpotent elements, then
so are these hulls.

1. Introduction. An $l$-group $G$ is said to be representable if it can be
embedded into a cardinal product of totally ordered groups, projectable
if each polar subgroup of the form $g^n$ for $g \in G$ is a cardinal summand of
$G$, and strongly projectable if every polar subgroup of $G$ is a cardinal
summand. If $G$ is projectable then each polar subgroup of $G$ is a normal
subgroup and so $G$, and hence each $l$-subgroup of $G$, is representable [4,
Theorem 1.8]. Conversely, if $G$ is representable then $G$ is a dense $l$-
subgroup of its orthocompletion [2, pp. 116, 125] which is strongly
projectable.

If $G$ is a large $l$-subgroup of a strongly projectable (projectable) $l$-group
$K$ then the intersection $H$ of all strongly projectable (projectable) $l$-
subgroups of $K$ containing $G$ is strongly projectable (projectable), and Conrad
[5] has shown that $H$ is the unique (up to $l$-isomorphism extending the
identity map of $G$) minimal strongly projectable (projectable) essential
extension of $G$, and called $H$ the sp-hull (p-hull) of $G$. In this note we wish
to give direct limit representations of these hulls in a fashion similar to
that in which Conrad [3, pp. 455–457] has represented the orthocompletion
of $G$. References containing related material are [1], [2], [5], [7].

NOTATION. We use the notation and terminology of the general
references [4], [6] unless otherwise specified. $G$ will always denote a

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representable l-group (written additively, but not assumed to be abelian). For a subset \( S \) of \( G \), \( S' \) denotes the polar of \( S \), and \( \mathcal{P}(G) \) the complete Boolean algebra of polar subgroups of \( G \). \( \prod A_i \) denotes the cardinal product of the l-groups \( A_i \) and \( A \oplus B \) denotes the cardinal sum of the l-groups \( A, B \).

2. Representation of the hulls. The following discussion of the direct limit of a directed system of l-groups and l-isomorphisms and construction of the orthocompletion of \( G \) was given by Conrad in [3], but is repeated here for completeness.

Let \( \Lambda \) be a lower directed set and suppose \( \{ G_\alpha, \Pi_{\alpha \beta}, \Lambda \} \) is a lower directed system of l-groups and l-isomorphisms (i.e., if \( \alpha \leq \beta \) then \( \Pi_{\alpha \beta} \) is an l-isomorphism of \( G_\alpha \) into \( G_\beta \) where the usual transitive laws are satisfied). If \( P \) is the set product of the \( G_\alpha \), let \( L \) denote the collection of all \( t \in P \) such that if \( \alpha \leq \beta \) then \( t_\alpha \neq 0 \) or \( t_\beta = 0 \) implies \( t_\alpha \Pi_{\alpha \beta} = t_\beta \), and \( t_\alpha = 0 \) and \( t_\beta \neq 0 \) implies \( t_\beta \notin \Pi_{\alpha \beta} \).

Let \( 0 \) denote the zero element of \( P \), and for \( \theta \neq k, t \in L \) select \( \alpha \) such that \( k_\alpha \neq 0 \neq t_\alpha \); then \( k + t \) is defined by \( (k + t)_\alpha = k_\alpha + t_\alpha \) (each nonzero component of a nonzero element of \( L \) completely determines that element). Also call \( t \in L \) positive if \( t = 0 \) or \( t_\alpha > 0 \) for some \( \alpha \). Then \( L \) is an l-group and the direct limit of the directed system above; if \( t \in L \) and \( t_\alpha \neq 0 \) then \( (t \vee 0)_\alpha = t_\alpha \vee 0 \).

Call a maximal disjoint subset of \( \mathcal{P}(G) \) a partition and let \( D(G) \) denote the set of all partitions of \( \mathcal{P}(G) \). If \( \mathcal{A} \) and \( \mathcal{C} \) are partitions then we write \( \mathcal{A} \leq \mathcal{C} \) (\( \mathcal{A} \) refines \( \mathcal{C} \)) if \( A \in \mathcal{A} \) implies \( A \subseteq C \) for some \( C \in \mathcal{C} \). If \( \mathcal{B} \in D(G) \), let \( \mathcal{D} = \{ C \cap B \neq 0 \mid C \in \mathcal{C}, B \in \mathcal{B} \} \) (the intersection of \( \mathcal{B} \) and \( \mathcal{C} \)); then \( \mathcal{D} \leq \mathcal{B}, \mathcal{C} \) and so \( D(G) \) is a lower directed set. If \( C \in \mathcal{C} \) and \( \{ A_\gamma \}_\Gamma \) is the collection of all elements of \( \mathcal{A} \) included in \( C \) then \( C = \bigcap A_\gamma \) and so the natural map of \( G[C'] \) into \( \prod \{ G[A_\gamma] \mid \gamma \in \Gamma \} \) is an l-isomorphism. In this way a natural l-isomorphism \( \Pi_{\mathcal{A}, \mathcal{C}} \) of \( G_{\mathcal{C}} = \prod \{ G[C] \mid C \in \mathcal{C} \} \) into \( G_{\mathcal{C}} \) is determined, and \( \{ G_{\mathcal{C}}, \Pi_{\mathcal{A}, \mathcal{C}}, D(G) \} \) is then a directed system; let its direct limit be \( L \). If \( t \in L \) and \( \mathcal{C} \in D(G) \) we write \( t_\mathcal{C} = (\cdots, t(C) + C', \cdots) \) where \( C \in \mathcal{C} \) and \( t(C) \in G \) (the \( t(C) \) are only determined mod \( C' \), of course). If \( g \in G \) and \( \mathcal{B} \in D(G) \) let \( \tilde{g}(B) = g \) for all \( B \in \mathcal{B} \); then the map \( g \rightarrow \tilde{g} \) is an l-isomorphism of \( G \) onto a dense l-subgroup \( \tilde{G} \) of \( L \). \( L \) is the orthocompletion of \( G \) in the sense of Bernau [2] (see [3, pp. 455–457]).

Now to construct the sp-hull of \( G \) let \( F(G) \) denote the set of all finite elements of \( D(G) \) and consider the directed system \( \{ G_{\mathcal{C}}, \Pi_{\mathcal{A}, \mathcal{C}}, F(G) \} \). If \( S \) is the direct limit of this system then we have a natural l-isomorphism \( g \rightarrow \tilde{g} \) of \( G \) into \( S \) given by \( \tilde{g}(B) = g \) for all \( B \in \mathcal{B} \) and all \( \mathcal{B} \in F(G) \), and \( \tilde{G} \) is a dense l-subgroup of \( S \). Therefore the map \( Q \rightarrow \mathcal{Q} \cap \tilde{G} \) determines a natural Boolean isomorphism of \( \mathcal{P}(S) \) onto \( \mathcal{P}(G) \) [3, p. 455], and so if
$P \in \mathcal{P}(G)$ and $\ast$ denotes the polar operation in $S$, then $\mathcal{G} \cap \mathcal{P}^{**}=\mathcal{P}$ (henceforth we will identify $G$ and $\mathcal{G}$).

If $\theta \neq f \in S$, $P \in \mathcal{P}(G)$, and $\mathcal{B} \in F(G)$ refines $\{P, P'\}$ then $f \in P^*$ if and only if $f(B) \equiv 0 \mod B'$ for all $B \in \mathcal{B}$ included in $P$. Similarly, $f \in P^{**}$ if and only if $f(B) \equiv 0 \mod B'$ for each $B \in \mathcal{B}$ included in $P'$. The proofs of these facts are routine.

**THEOREM A.** $S$ is the sp-hull of $G$.

**Proof.** Let $Q \in \mathcal{P}(S)$ and $\theta \neq t \in S$; $Q$ is of the form $P^{**}$ for a unique $P \in \mathcal{P}(G)$. Pick $B \in F(G)$ such that $B$ refines $\{P, P'\}$ and $\theta \in B$, and define $k \in S$ by specifying that, for all $B \in B$, $k(B)=t(B)$ if $P \subset P$, and $k(B)=0$ otherwise. Then $k \in P^{**} \subset Q$ and $t-k \in P^* \subset Q^*$, so that $t=(t-k)+(k \ast t) \in Q \oplus Q^*$. Therefore $S$ is a strongly projectable essential extension of $G$.

Now suppose that $G \subseteq H \subseteq S$ where $H$ is strongly projectable and let $\perp$ denote the polar operation in $H$. Assume that $t \in H$, $Q \in \mathcal{P}(H)$, and write $t=t_1+t_2$ in $Q \oplus Q^*=H$ and $t=k_1+k_2$ in $Q^{**} \oplus Q^*=S$. Since $Q \subseteq Q^{**}$ and $Q^\perp \subseteq Q^*$ it follows that $t_1=k_1$ and $t_2=k_2$; thus the projections $t_1$ of $t$ into $Q$ in $H$ and $k_1$ of $t$ into $Q^{**}$ in $S$ are identical.

Now let $\theta \neq t \in S$ with $t_\perp=(g_1+C_1, \cdots, g_n+C_n)$ and let $t_j \in S$ be determined by $t_j(C_j)=g_j$ and $t_j(C_i)=0$ for $i \neq j$, $1 \leq j \leq n$. Then $t_j$ is the projection of $g_j$ into $C_j^{**}$, and since $g_j \in H$ it follows as discussed in the last paragraph that $t_j$ is also the projection of $g_j$ into $C_j^\perp$. Thus $t_j \in H$ and so $t=t_1+\cdots+t_n \in H$. Hence $H=S$, and so $S$ is a minimal strongly projectable essential extension of $G$.

**Remark.** If $A$ is an $f$-ring then each polar subgroup of $A$ is an ideal and so any directed system of the $A_{\perp}$ and $\Pi_{\perp}$ is a system of $f$-rings and $l$-ring isomorphisms. Thus the direct limit of such a system is also an $f$-ring. If $A$ has no nonzero nilpotent elements then $A/P$ also has none for each polar subgroup $P$ of $A$, and hence such a direct limit also has this property.

In particular, the sp-hull $S$ of $A$ is an $f$-ring (with no nonzero nilpotent elements if $A$ has none), and the map $g \rightarrow \bar{g}$ is an $l$-ring isomorphism of $A$ into $S$.

The p-hull of $G$ can be constructed in an analogous (but somewhat more complicated) fashion. Here let $T$ denote the sublattice of $\mathcal{P}(G)$ generated by all polars of the form $g^\star$ or $g'$ for $g \in G$ and let $T(G)$ be the set of all $C \in F(G)$ which are subsets of $T$. This gives rise to a directed system $\{G_{\perp}, \Pi_{\perp}, T(G)\}$, the direct limit of which we denote by $T$. Again we have a natural embedding $g \rightarrow \bar{g}$ given by $\bar{g}(C)=g$ for all $C \in C$ and all $C \in T(G)$ and we denote the polar operations in $G$ and $T$ by $'$ and $\ast$ respectively. We identify $G$ and $\mathcal{G}$. 

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**Lemma.** If $\theta \neq k \in T$ with $0 \neq k_\theta = (\cdots, k_j + C_j, \cdots)$ then $k^\ast = \vee (C_j^\ast \cap k_j^\ast)$ where the sup occurs in $P(T)$.

**Proof.** Without loss of generality $k > \theta$. Assume first that $G = \{C_j\}_{j=1}^n$ and $0 \neq k_\theta = (g + C_1, 0, \cdots, 0)$. If $D$ is the intersection of $G$ and $\{g^\ast, g'\}$ then $0 \neq k_D = (g + (C_1 \cap g')^\ast, 0, \cdots, 0)$ and so $k \in (C_1 \cap g')^\ast \cap g^\ast$. We wish to show that if $0 < t \in C_1^\ast \cap g^\ast$ then $t \land k > \theta$; then it will follow that $k^\ast = C_1^\ast \cap g^\ast$, as desired.

Consider $0 < t \in C_1^\ast \cap g^\ast$ and let $t_D > 0$ for some $D$ refining $D$. Then $t(B) = 0 \mod B'$ when $B \in D$ is not included in $C_1 \cap g^\ast$, and if $B \subseteq C_1 \cap g^\ast$ and $g \notin B'$ then $g + B'$ is a weak unit for $G/B'$. Therefore if $t(B), g \notin B'$ for some $B \in D$ then $t_{\theta \land k} > 0$ and so $0 < t \land k$ as desired. Suppose by way of contradiction that for all $B \in D$, $g \notin B'$ implies $t(B) \in B'$ then $g_{\theta \land t_{\theta \land k}} = 0$ and so $t \in g^\ast$. Hence $0 < t \in g^\ast \cap g^\ast = \theta$, a contradiction. It follows then that $t \land k > \theta$ and so $k^\ast = C_1^\ast \cap g^\ast$.

In the general case $0 < k$ and $0 < k_\theta = (k_1 + C_1, \cdots, k_n + C_n)$. If $0 \leq t_j \in T$ is determined by $(t_j)_\theta = (0, \cdots, 0, k_j + C_j, 0, \cdots, 0)$, then

$t_j^\ast = k_j^\ast \cap C_j^\ast$ and $k = \vee t_j$

so that $k^\ast = \vee t_j^\ast = \vee (k_j^\ast \cap C_j^\ast)$.

**Theorem B.** $T$ is the $p$-hull of $G$.

**Proof.** Suppose $0 \neq k, f \in T$ and $0 \neq k_\theta = (k_1 + C_1, \cdots, k_n + C_n)$. Then by the lemma $k^\ast = \vee (C_j^\ast \cap k_j^\ast)$, and so if $k^\ast \cap G = P \in P(G)$ then $P = \vee (G \cap C_j^\ast \cap k_j^\ast) = \vee (C_j \cap k_j^\ast)$. Thus $P \in T$ and hence $\{P, P'\} \in T(G)$. Pick $B \in T(G)$ refining $\{P, P'\}$ such that

$0 \neq f_{\theta \land B} = (f_1 + B_1, \cdots, f_m + B_m)$,

and let $t \in T$ be specified by $t(B_j) = f_j$ if $B_j \subseteq P$ and $t(B_j) = 0$ otherwise. Then $f = t + (f - t) \in P^\ast \oplus P^\ast = k^\ast \oplus k^\ast$, and so it follows that $T$ is projectable.

Now suppose that $G \subseteq H \subseteq T$ where $H$ is projectable and let $\perp$ denote the polar operation in $H$. If $h \in H$ then $h^\perp \subseteq h^\ast$ and $h^\perp \subseteq h^\perp \perp \ast = h^\ast$ [5, (5i)]; thus if $k \in H$ then the projections of $k$ into $h^\perp \perp$ in $H$ and into $h^\ast$ in $T$ coincide.

Assume now that $0 < t \in T$ with $t_\theta > 0$.

**Case 1.** Suppose first that $t_\theta = (g + C_1, 0, \cdots, 0)$ where without loss of generality $0 < g \notin C_1$. Since $C_i \subseteq T$ we have $C_1 = \bigvee_I \bigwedge_J S_{i,j}$ where $I$ and $J$ are finite sets and $S_{i,j}$ has the form $g_{i,j}$ or $g_{i,j}'$ for some $g_{i,j} \in G$. Let $D$ be the intersection of all the partitions $\{S_{i,j}, S_{i,j}'\}$; we claim that $D$ refines $\{C_1, C_1'\}$. For if $B \in D$ and $B \notin C_1$, then for each $i$ there is a $j_i$ such that $B \notin S_{i,j_i}$; Then $B \subseteq S_{i,j_i}'$ and so $B \subseteq \bigwedge_I \bigvee_J S_{i,j_i}' = C_1$. 

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Now let \( t_{ij} \in T \) be determined by \( t_{ij}(B) = g \) if \( B \subseteq S_{ij} \) and \( t_{ij}(B) = 0 \) otherwise, and let \( k = \bigvee_{i} \Lambda J t_{ij} \). If \( k(B) \neq 0 \mod|B'| \) then \( B \subseteq C_1 \), while if \( B \subseteq C_1 \) then \( B \neq C_1 \) and so it follows that \( B \subseteq \bigwedge J S_{ij} \) for some \( i \), and hence \( k(B) = g = t(B) \). Therefore \( t_{ij} = k_{ij} \) and so \( t = \bigvee_{i} \Lambda J t_{ij} \). Since \( t_{ij} \) is the projection of \( g \in H \) into \( S_{ij}^{**} = g_{ij}^{**} \) or \( g_{ij}^{*} \), we have \( t_{ij} \in H \); thus \( t \in H \).

**Case 2.** In the general case we have \( t = t_{1} + \cdots + t_{n} \) where the \( t_{ij} \) have the form considered in Case 1. Thus \( t_{j} \in H \) and so \( t \in H \). Therefore \( H = T \), and so \( T \) is a minimal projectable essential extension of \( G \).

**Remark.** It follows as before that if \( A \) is an \( f \)-ring, or an \( f \)-ring with no nonzero nilpotent elements, then so is the \( p \)-hull of \( A \).

**References**


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