L^p-CONVOLUTION OPERATORS SUPPORTED BY SUBGROUPS

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ABSTRACT. Let G be a compact nonabelian group and H be a closed subgroup of G. Then H is a set of spectral synthesis for the Fourier algebra A(G) (and indeed for A^*(G), 1 ≤ p < ∞). For 1 ≤ p < ∞, each L^p(G)-multiplier T corresponds to a L^p(H)-multiplier S by the rule (Tf)|H=S(f)|H, f ∈ A(G), if and only if the support of T is contained in H.

Let G be a compact nonabelian group and Ĝ its dual. We denote the Fourier algebra by A(G) and its dual by L^∞(Ĝ). We will use the notation from our book [1].

Let φ ∈ L^∞(Ĝ), then the support of φ, denoted by spt φ, is defined to be the intersection of the sets {K ⊆ G: K is compact and (f, φ) = 0 whenever the support of f ⊆ G\K, f ∈ A(G)} [1, p. 94]. For f ∈ C(G), spt f denotes the usual support of f. For u a bounded Borel function on G, define ̂u by ̂u(x) = u(x^{-1}), x ∈ G.

Let E be a closed subset of G. The set E is called a set of spectral synthesis for A(G) provided whenever f ∈ A(G), f(x) = 0 for x ∈ E, and ε > 0, there exists g ∈ A(G) with g = 0 on a neighborhood of E and ∥f − g∥_A < ε. We will show that closed subgroups H of G are sets of spectral synthesis for A(G). Our proof is adapted from [3] where the result is given for H normal. Henceforth H will be a fixed closed subgroup of G, with normalized Haar measure m_H.

PROPOSITION 1. Let f ∈ A(G), f = 0 on H, and ε > 0. Then there exists a neighborhood W of the identity e of G such that if u is a nonnegative bounded Borel function on W, and \int_G u(x) d m_G(x) = 1, then \|f − ̂f̂u\|_A ≤ ε.

PROOF. Since translation is continuous in A(G) [1, p. 91], there exists a neighborhood W of e such that if y ∈ W, then \|f − R(y)f\|_A ≤ ε (R(y)f(x) = f(xy), x, y ∈ G).

Received by the editors October 12, 1971.

AMS 1970 subject classifications. Primary 43A22, 43A45.

Key words and phrases. Fourier algebra, spectral synthesis, L^p-multiplier.

1 This research was supported in part by NSF contract no. GP-19852.

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Thus
\[ \| f - f * \mathring{\tau} \|_A = \sup \left\{ \int_G (f - f * \mathring{\tau}) g \, dm_G \colon g \in L^1(G), \| \mathring{\tau} \|_\infty \leq 1 \right\} \]

(see [1, p. 92])
\[ = \sup \left\{ \int_G \int_W (f(x) - R(y) f(x)) u(y) \, dm_G(y) g(x) \, dm_G(x) \colon g \in L^1(G), \| \mathring{\tau} \|_\infty \leq 1 \right\} \]
\[ \leq \varepsilon. \quad \square \]

The proof of the following proposition was shown to us by our colleague R. E. Stong.

**Proposition 2.** Let \( W \) be a neighborhood of \( e \). There exists a non-negative continuous function \( w \) on \( G \) with \( \text{spt} \ w \subseteq W \), such that the function \( \pi w = m_H * w \) is equal to 1 on \( HW' \) (\( W' \) a neighborhood of \( e \)).

**Proof.** Let \( h_1, \ldots, h_n \in H \) be such that \( \bigcup_{i=1}^n h_i W \supseteq H \). Choose a neighborhood \( W' \) of \( e \) with \( H \subseteq \text{cl}(HW') \subseteq \bigcup_{i=1}^n h_i W \) (\( \text{cl} \) denotes closure). Let \( \phi_1, \ldots, \phi_n \) be a partition of unity subordinate to the cover \( \{h_1 W, \ldots, h_n W\} \) (\( \text{spt} \ \phi_i \subseteq h_i W \), \( i = 1, \ldots, n \)) such that \( \sum_{i=1}^n \phi_i(x) = 1 \) for \( x \in HW' \).

Let \( w(x) = \sum_{i=1}^n \phi_i(h_i x), \ x \in G \). Then \( \text{spt} \ w \subseteq W \). Finally, let \( x \in HW' \); then
\[ \pi w(x) = (m_H * w)(x) = \int_H w(hx) \, dm_H(h) \]
\[ = \int_H \sum_{i=1}^n \phi_i(h_i hx) \, dm_H(h) \]
\[ = \int_H \sum_{i=1}^n \phi_i(hx) \, dm_H(h) = \int_H 1 \, dm_H(h) = 1. \quad \square \]

**Theorem 3.** Let \( H \) be a closed subgroup of \( G \). Then \( H \) is a set of spectral synthesis for \( A(G) \).

**Proof.** Let \( f \in A(G) \), \( f = 0 \) on \( H \), and \( \varepsilon > 0 \). Let \( W \) be as in Proposition 1. Now choose \( w, \pi w, \) and \( W' \) as in Proposition 2. Since \( f = 0 \) on \( H \), there exists a neighborhood \( V \) of \( e \) such that \( |f^2(x)| \leq \varepsilon^2 / \| w^2 \|_\infty \) for \( x \in HV \).

Now choose neighborhoods \( U, U' \) of \( e \) such that \( U' H U \subseteq HV \cap HW' \) and \( m_G(U' H U) \leq 4 m_G(HU) \).

Let \( u \) and \( v \) be bounded Borel functions on \( G \) defined by
\[ u(x) = (m_G(HU))^{-1} w(x), \quad \text{if } x \in HU, \]
\[ = 0, \quad \text{if } x \notin HU, \]
and

\[ v(x) = f(x), \quad \text{if } x \in U'HU, \]
\[ = 0, \quad \text{if } x \notin U'HU. \]

Write \( f = f*\check{u} + (f-v)*\check{u} + v*\check{u} \), and let \( g = (f-v)*\check{u} \) and \( h = v*\check{u} \). Note that \( f*\check{u} \), \( g \), \( h \in A(G) = L^2(G) \ast L^2(G) \) [1, p. 92]. We will show that \( \|f - g\|_A \leq 3\varepsilon \) and \( g = 0 \) on a neighborhood of \( H \).

Let \( x \in U'H \), then

\[ g(x) = (f - v) * \check{u}(x) = \int_G (f - v)(xy)u(y) \, dm_G(y) = \int_{HU} (f(xy) - v(xy))u(y) \, dm_G(y) = 0 \]

since \( xy \in U'HHU = U'HU \). Thus \( \text{spt } g \subseteq H \).

Now \( \|f - f*\check{u}\|_A \leq \varepsilon \) since \( u \) is a nonnegative bounded Borel function on \( G \), \( \text{spt } u \subseteq \text{spt } w \subseteq W \), and

\[ \int_G u(x) \, dm_G(x) = \frac{1}{m_G(HU)} \int_{HU} w(x) \, dm_G(x) \]
\[ = \frac{1}{m_G(HU)} \int_{HU} \pi w(Hx) \, d\omega(Hx) = 1 \]

(\( \omega \in M(G/H) \) is the unique normalized measure such that \( \int_{G/H} R(x)f \, d\omega = \int_{G/H} f \, d\omega \), where \( R(x)f(Hy) = f(Hyx) \), \( x, y \in G, f \in C(G/H) \); see [1, p. 101]).

It remains to show that \( \|h\|_A \leq 2\varepsilon \). Now \( \|h\|_A \leq \|v\|_2 \|u\|_2 \) and

\[ \|\check{u}\|_2^2 = \|u\|_2^2 = \int_G u^2(x) \, dm_G(x) \leq \|w^2\|_\infty / m_G(HU). \]

Thus \( \|\check{u}\|_2 \leq (\|w^2\|_\infty / m_G(HU))^{1/2} \). Also

\[ \|v\|_2^2 = \int_{U'HU} |f^2(x)| \, dm_G(x) \leq (\varepsilon^2/\|w^2\|_\infty) m_G(U'HU) \leq 4\varepsilon^2 m_G(HU)/\|w^2\|_\infty. \]

Thus \( \|v\|_2 \leq 2\varepsilon (m_G(HU)/\|w^2\|_\infty)^{1/2} \), and so \( \|h\|_A \leq 2\varepsilon \). \( \Box \)

Remark. Let \( 1 \leq p < \infty \) and \( A^p(G) \) the predual of \( M^p(G) \), the \( L^p(G) \)-multipliers (see [2, p. 500]). If \( H \) is a closed subgroup of the compact group \( G \), then \( H \) is a set of spectral synthesis for \( A^p(G) \). The proof is the same as the proof of Theorem 3 with only slight modifications.

Corollary 4. Suppose \( H \) is a closed subgroup of a compact group \( G \) and \( T \) is an \( L^p(G) \)-convolution operator. Then \( T \) corresponds to an
$L^p(H)$-convolution operator $S$ by the rule $(Tf)|H = S(f|H)$, $f \in A(G)$, if and only if $\text{spt } T \subseteq H$. This correspondence is an isometry.

**Proof.** This follows immediately from the result of Herz [3, p. 317] that the restriction map $f \to f|H$ from $A^p(G)$ to $A^p(H)$ is onto. □

**Remark.** For $G$ a locally compact abelian group and $H$ a closed subgroup, the analogous result of Corollary 4 has been shown by S. Saeki [5]. For $G$ a locally compact group and $H$ a compact normal subgroup, the analogous result to Corollary 4 has been shown by C. Herz ([3], [4]).

**Remark.** Let $G$ be a compact group and $H$ a normal closed subgroup. If $T \in M^p(G)$ such that $Tf = T(m_H * f)$, $f \in A(G)$, then there exists $S \in M^p(G/H)$ such that $Sf = Tf$ for $f \in A(G/H) = m_H * A(G)$ [1, p. 106], and $\|S\| = \|T\|$. Conversely, if $S \in M^p(G/H)$, then there exists $T \in M^p(G)$ defined by $Tf = S(m_H * f)$, $f \in A(G)$, and $\|T\| = \|S\|$.  

**Bibliography**


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