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L^p-CONVOLUTION OPERATORS SUPPORTED BY SUBGROUPS[1]

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ABSTRACT. Let G be a compact nonabelian group and H be a closed subgroup of G. Then H is a set of spectral synthesis for the Fourier algebra $A(G)$ (and indeed for $A^p(G)$, $1 \leqq p < \infty$). For $1 \leqq p < \infty$, each $L^p(G)$-multiplier T corresponds to a $L^p(H)$-multiplier S by the rule $(Tf)|H = S(f|H)$, $f \in A(G)$, if and only if the support of T is contained in H.

Let G be a compact nonabelian group and $\hat{G}$ its dual. We denote the Fourier algebra by $A(G)$ and its dual by $\mathscr{L}^\infty(\hat{G})$. We will use the notation from our book [**1**].

Let $\phi \in \mathscr{L}^\infty(\hat{G})$, then the support of ϕ, denoted by spt ϕ, is defined to be the intersection of the sets $\{K \subset G : K$ is compact and $\langle f, \phi \rangle = 0$ whenever the support of $f \subset G \backslash K, f \in A(G)\}$ [**1**, p. 94]. For $f \in C(G)$, spt f denotes the usual support of f. For u a bounded Borel function on G, define $\check{u}$ by $\check{u}(x) = u(x^{-1})$, $x \in G$.

Let E be a closed subset of G. The set E is called a set of spectral synthesis for $A(G)$ provided whenever $f \in A(G)$, $f(x) = 0$ for $x \in E$, and $\varepsilon > 0$, there exists $g \in A(G)$ with $g = 0$ on a neighborhood of E and $\|f - g\|_A < \varepsilon$. We will show that closed subgroups H of G are sets of spectral synthesis for $A(G)$. Our proof is adapted from [**3**] where the result is given for H normal. Henceforth H will be a fixed closed subgroup of G, with normalized Haar measure m_H.

PROPOSITION 1. *Let $f \in A(G)$, $f = 0$ on H, and $\varepsilon > 0$. Then there exists a neighborhood W of the identity e of G such that if u is a nonnegative bounded Borel function on W, and $\int_G u(x)\, dm_G(x) = 1$, then $\|f - f * \check{u}\|_A \leqq \varepsilon$.*

PROOF. Since translation is continuous in $A(G)$ [**1**, p. 91], there exists a neighborhood W of e such that if $y \in W$, then $\|f - R(y)f\|_A \leqq \varepsilon$ ($R(y)f(x) = f(xy)$, $x, y \in G$).

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Thus

$$\|f-f*\check{u}\|_A = \sup\left\{\left|\int_G (f-f*\check{u})g\,dm_G\right| : g\in L^1(G),\ \|\hat{g}\|_\infty \leqq 1\right\}$$

(see [1, p. 92])

$$= \sup\left\{\left|\int_G\int_W (f(x)-R(y)f(x))u(y)\,dm_G(y)g(x)\,dm_G(x)\right| : g\in L^1(G),\ \|\hat{g}\|_\infty \leqq 1\right\}$$

$$\leqq \varepsilon. \quad \square$$

The proof of the following proposition was shown to us by our colleague R. E. Stong.

PROPOSITION 2. *Let W be a neighborhood of e. There exists a non-negative continuous function w on G with* spt $w\subset W$, *such that the function $\pi w=m_H*w$ is equal to* 1 *on HW' (W' a neighborhood of e).*

PROOF. Let $h_1,\cdots,h_n\in H$ be such that $\bigcup_{i=1}^n h_iW\supset H$. Choose a neighborhood W' of e with $H\subset \mathrm{cl}(HW')\subset\bigcup_{i=1}^n h_iW$ (cl denotes closure). Let $\phi_1,\cdots,\phi_n$ be a partition of unity subordinate to the cover $\{h_1W,\cdots,h_nW\}$ (spt $\phi_i\subset h_iW$, $i=1,\cdots,n$) such that $\sum_{i=1}^n\phi_i(x)=1$ for $x\in HW'$.

Let $w(x)=\sum_{i=1}^n\phi_i(h_ix)$, $x\in G$. Then spt $w\subset W$. Finally, let $x\in HW'$; then

$$\pi w(x) = (m_H*w)(x) = \int_H w(hx)\,dm_H(h)$$
$$= \int_H\sum_{i=1}^n \phi_i(h_ihx)\,dm_H(h)$$
$$= \int_H\sum_{i=1}^n \phi(hx)\,dm_H(h) = \int_H 1\,dm_H(h) = 1. \quad \square$$

THEOREM 3. *Let H be a closed subgroup of G. Then H is a set of spectral synthesis for $A(G)$.*

PROOF. Let $f\in A(G)$, $f=0$ on H, and $\varepsilon>0$. Let W be as in Proposition 1. Now choose w, πw, and W' as in Proposition 2. Since $f=0$ on H, there exists a neighborhood V of e such that $|f^2(x)|\leqq\varepsilon^2/\|w^2\|_\infty$ for $x\in HV$. Now choose neighborhoods U, U' of e such that $U'HU\subset HV\cap HW'$ and $m_G(U'HU)\leqq 4m_G(HU)$.

Let u and v be bounded Borel functions on G defined by

$$u(x) = (m_G(HU))^{-1}w(x), \quad \text{if } x\in HU,$$
$$= 0, \quad \text{if } x\notin HU,$$

and

$$\begin{aligned} v(x) &= f(x), \quad \text{if } x \in U'HU, \\ &= 0, \qquad \text{if } x \notin U'HU. \end{aligned}$$

Write $f = f - f*\check{u} + (f-v)*\check{u} + v*\check{u}$, and let $g = (f-v)*\check{u}$ and $h = v*\check{u}$. Note that $f*\check{u}$, g, $h \in A(G) = L^2(G)*L^2(G)$ [**1**, p. 92]. We will show that $\|f-g\|_A \leqq 3\varepsilon$ and $g=0$ on a neighborhood of H.

Let $x \in U'H$, then

$$\begin{aligned} g(x) = (f-v) * \check{u}(x) &= \int_G (f-v)(xy)u(y)\, dm_G(y) \\ &= \int_{HU} (f(xy) - v(xy))u(y)\, dm_G(y) = 0 \end{aligned}$$

since $xy \in U'HHU = U'HU$. Thus $\operatorname{spt} g \cap H = \varnothing$.

Now $\|f - f*\check{u}\|_A \leqq \varepsilon$ since u is a nonnegative bounded Borel function on G, $\operatorname{spt} u \subset \operatorname{spt} w \subset W$, and

$$\begin{aligned} \int_G u(x)\, dm_G(x) &= \frac{1}{m_G(HU)} \int_{HU} w(x)\, dm_G(x) \\ &= \frac{1}{m_G(HU)} \int_{HU} \pi w(Hx)\, d\omega(Hx) = 1 \end{aligned}$$

(where $\omega \in M(G/H)$ is the unique normalized measure such that $\int_{G/H} R(x) f\, d\omega = \int_{G/H} f\, d\omega$, where $R(x)f(Hy) = f(Hyx)$, $x, y \in G, f \in C(G/H)$; see [**1**, p. 101]).

It remains to show that $\|h\|_A \leqq 2\varepsilon$. Now $\|h\|_A \leqq \|v\|_2\, \|u\|_2$ and

$$\|\check{u}\|_2^2 = \|u\|_2^2 = \int_G u^2(x)\, dm_G(x) \leqq \|w^2\|_\infty / m_G(HU).$$

Thus $\|\check{u}\|_2 \leqq (\|w^2\|_\infty / m_G(HU))^{1/2}$. Also

$$\begin{aligned} \|v\|_2^2 &= \int_{U'HU} |f^2(x)|\, dm_G(x) \\ &\leqq (\varepsilon^2/\|w^2\|_\infty) m_G(U'HU) \leqq 4\varepsilon^2 m_G(HU)/\|w^2\|_\infty. \end{aligned}$$

Thus $\|v\|_2 \leqq 2\varepsilon(m_G(HU)/\|w^2\|_\infty)^{1/2}$, and so $\|h\|_A \leqq 2\varepsilon$. □

REMARK. Let $1 \leqq p < \infty$ and $A^p(G)$ the predual of $M^p(G)$, the $L^p(G)$-multipliers (see [**2**, p. 500]). If H is a closed subgroup of the compact group G, then H is a set of spectral synthesis for $A^p(G)$. The proof is the same as the proof of Theorem 3 with only slight modifications.

COROLLARY 4. *Suppose H is a closed subgroup of a compact group G and T is an $L^p(G)$-convolution operator. Then T corresponds to an*

$L^p(H)$-convolution operator S by the rule $(Tf)|H=S(f|H)$, $f\in A(G)$, if and only if spt $T\subset H$. *This correspondence is an isometry.*

PROOF. This follows immediately from the result of Herz [3, p. 317] that the restriction map $f\mapsto f|H$ from $A^p(G)$ to $A^p(H)$ is onto. □

REMARK. For G a locally compact abelian group and H a closed subgroup, the analogous result of Corollary 4 has been shown by S. Saeki [5]. For G a locally compact group and H a compact normal subgroup, the analogous result to Corollary 4 has been shown by C. Herz ([3], [4]).

REMARK. Let G be a compact group and H a normal closed subgroup. If $T\in M^p(G)$ such that $Tf=T(m_H*f)$, $f\in A(G)$, then there exists $S\in M^p(G/H)$ such that $Sf=Tf$ for $f\in A(G/H)=m_H*A(G)$ [1, p. 106], and $\|S\|=\|T\|$. Conversely, if $S\in M^p(G/H)$, then there exists $T\in M^p(G)$ defined by $Tf=S(m_H*f)$, $f\in A(G)$, and $\|T\|=\|S\|$.

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