$L^p$-CONVOLUTION OPERATORS SUPPORTED BY SUBGROUPS

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ABSTRACT. Let $G$ be a compact nonabelian group and $H$ be a closed subgroup of $G$. Then $H$ is a set of spectral synthesis for the Fourier algebra $A(G)$ (and indeed for $A^*(G)$, $1 \leq p < \infty$). For $1 \leq p < \infty$, each $L^p(G)$-multiplier $T$ corresponds to a $L^p(H)$-multiplier $S$ by the rule $(Tf)|H = S(f|H)$, $f \in A(G)$, if and only if the support of $T$ is contained in $H$.

Let $G$ be a compact nonabelian group and $\hat{G}$ its dual. We denote the Fourier algebra by $A(G)$ and its dual by $L^\infty(\hat{G})$. We will use the notation from our book [1].

Let $\phi \in L^\infty(\hat{G})$, then the support of $\phi$, denoted by $\text{spt} \phi$, is defined to be the intersection of the sets $\{K \subseteq G : K$ is compact and $(f, \phi) = 0$ whenever the support of $f \subseteq G \setminus K, f \in A(G)\}$ [1, p. 94]. For $f \in C(G)$, $\text{spt} f$ denotes the usual support of $f$. For $\mu$ a bounded Borel function on $G$, define $\hat{\mu}$ by $\hat{\mu}(x) = \mu(x^{-1}), x \in G$.

Let $F$ be a closed subset of $G$. The set $F$ is called a set of spectral synthesis for $A(G)$ provided whenever $f \in A(G)$, $f(x) = 0$ for $x \in E$, and $\varepsilon > 0$, there exists $g \in A(G)$ with $g = 0$ on a neighborhood of $E$ and $\|f - g\|_A < \varepsilon$. We will show that closed subgroups $H$ of $G$ are sets of spectral synthesis for $A(G)$. Our proof is adapted from [3] where the result is given for $H$ normal. Henceforth $H$ will be a fixed closed subgroup of $G$, with normalized Haar measure $m_H$.

PROPOSITION 1. Let $f \in A(G), f = 0$ on $H$, and $\varepsilon > 0$. Then there exists a neighborhood $W$ of the identity $e$ of $G$ such that if $u$ is a nonnegative bounded Borel function on $W$, and $\int_G u(x) \, dm_G(x) = 1$, then $\|f - f\hat{\mu}\|_A \leq \varepsilon$.

PROOF. Since translation is continuous in $A(G)$ [1, p. 91], there exists a neighborhood $W$ of $e$ such that if $y \in W$, then $\|f - R(y)f\|_A \leq \varepsilon$ ($R(y)f(x) = f(xy), x, y \in G$).

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Thus
\[ \|f - f \ast \tilde{u}\|_A = \sup \left\{ \left\| \int_G (f - f \ast \tilde{u}) g \, dm_G \right\| : g \in L^1(G), \|\tilde{g}\|_\infty \leq 1 \right\} \]

(see [1, p. 92])

\[ = \sup \left\{ \int_G \int_W (f(x) - R(y)f(x))u(y) \, dm_G(y)g(x) \, dm_G(x) : g \in L^1(G), \|\tilde{g}\|_\infty \leq 1 \right\} \]

\[ \leq \varepsilon. \quad \square \]

The proof of the following proposition was shown to us by our colleague R. E. Stong.

**PROPOSITION 2.** Let \( W \) be a neighborhood of \( e \). There exists a non-negative continuous function \( w \) on \( G \) with \( \text{spt } w \subseteq W \), such that the function \( \pi w = m_H \ast w \) is equal to 1 on \( HW' \) (\( W' \) a neighborhood of \( e \)).

**Proof.** Let \( h_1, \ldots, h_n \in H \) be such that \( \bigcup_{i=1}^n h_iW \supset H \). Choose a neighborhood \( W' \) of \( e \) with \( H \subseteq \text{cl}(HW') \subseteq \bigcup_{i=1}^n h_iW \) (\( \text{cl} \) denotes closure). Let \( \phi_1, \ldots, \phi_n \) be a partition of unity subordinate to the cover \( \{h_1W, \ldots, h_nW\} \) (\( \text{spt } \phi_i \subseteq h_iW, i=1, \ldots, n \)) such that \( \sum_{i=1}^n \phi_i(x) = 1 \) for \( x \in HW' \).

Let \( w(x) = \sum_{i=1}^n \phi_i(h_ix), x \in G \). Then \( \text{spt } w \subseteq W \). Finally, let \( x \in HW' \); then

\[ \pi w(x) = (m_H \ast w)(x) = \int_H w(hx) \, dm_H(h) \]

\[ = \int_H \sum_{i=1}^n \phi_i(h_ix) \, dm_H(h) \]

\[ = \int_H \sum_{i=1}^n \phi(hx) \, dm_H(h) = \int_H 1 \, dm_H(h) = 1. \quad \square \]

**THEOREM 3.** Let \( H \) be a closed subgroup of \( G \). Then \( H \) is a set of spectral synthesis for \( A(G) \).

**Proof.** Let \( f \in A(G), f=0 \) on \( H \), and \( \varepsilon > 0 \). Let \( W \) be as in Proposition 1. Now choose \( w, \pi w, \) and \( W' \) as in Proposition 2. Since \( f=0 \) on \( H \), there exists a neighborhood \( V \) of \( e \) such that \( |f^2(x)| \leq \varepsilon^2 \|w^2\|_\infty \) for \( x \in HV \). Now choose neighborhoods \( U, U' \) of \( e \) such that \( U'HU \subseteq HV \cap HW' \) and \( m_G(U'HU) \leq 4m_G(HU) \).

Let \( u \) and \( v \) be bounded Borel functions on \( G \) defined by

\[ u(x) = (m_G(HU))^{-1}w(x), \quad \text{if } x \in HU, \]

\[ = 0, \quad \text{if } x \notin HU, \]
and
\[ v(x) = f(x), \quad \text{if } x \in U' H U, \]
\[ = 0, \quad \text{if } x \notin U' H U. \]

Write \( f = f - f \ast \tilde{u} + (f - v) \ast \tilde{u} + v \ast \tilde{u} \), and let \( g = (f - v) \ast \tilde{u} \) and \( h = v \ast \tilde{u} \). Note that \( f \ast \tilde{u} \), \( g \), \( h \in A(G) = L^2(G) \ast L^2(G) \) \( \{1, \text{p. } 92\} \). We will show that \( \|f - g\|_A \leq 3 \varepsilon \) and \( g = 0 \) on a neighborhood of \( H \).

Let \( x \in U' H \), then
\[ g(x) = (f - v) \ast \tilde{u}(x) = \int_{\text{H} \mathcal{U}} (f - v)(xy) u(y) \, dm_G(y) \]
\[ = \int_{\text{H} \mathcal{U}} (f(xy) - v(xy)) u(y) \, dm_G(y) = 0 \]
since \( xy \in U' H H H U = U' H U \). Thus \( \text{spt} \, g \subset H = \emptyset \).

Now \( \|f - f \ast \tilde{u}\|_A \leq \varepsilon \) since \( u \) is a nonnegative bounded Borel function on \( G \), \( \text{spt} \, u \subset \text{spt} \, w \subset W \), and
\[ \int_G u(x) \, dm_G(x) = \frac{1}{m_G(HU)} \int_{H U} w(x) \, dm_G(x) \]
\[ = \frac{1}{m_G(HU)} \int_{H U} \pi w(Hx) \, dw(Hx) = 1 \]
(\( \omega \in M(G/H) \) is the unique normalized measure such that \( \int_{G/H} R(X)f \, dw = \int_{G/H} f \, dw \), where \( R(x)f(Hy) = f(Hyx), x, y \in G, f \in C(G/H) \); see \( \{1, \text{p. } 101\} \)).

It remains to show that \( \|h\|_A \leq 2 \varepsilon \). Now \( \|h\|_A \leq \|v\|_A \|u\|_A \) and
\[ \|\tilde{u}\|_2^2 = \|u\|_A^2 = \int_G u^2(x) \, dm_G(x) \leq \|w^2\|_\infty / m_G(HU). \]

Thus \( \|\tilde{u}\|_2^2 \leq (\|w^2\|_\infty / m_G(HU))^{1/2} \). Also
\[ \|v\|_2^2 = \int_{U' H U} |f^2(x)| \, dm_G(x) \leq (\varepsilon^2 / \|w^2\|_\infty) m_G(U' H U) \leq 4 \varepsilon^2 m_G(HU) / \|w^2\|_\infty. \]
Thus \( \|v\|_2 \leq 2 \varepsilon(m_G(HU) / \|w^2\|_\infty)^{1/2} \), and so \( \|h\|_A \leq 2 \varepsilon \). \( \square \)

**Remark.** Let \( 1 \leq p < \infty \) and \( A^p(G) \) the predual of \( M^p(G) \), the \( L^p(G) \)-multipliers (see \( \{2, \text{p. } 500\} \)). If \( H \) is a closed subgroup of the compact group \( G \), then \( H \) is a set of spectral synthesis for \( A^p(G) \). The proof is the same as the proof of Theorem 3 with only slight modifications.

**Corollary 4.** Suppose \( H \) is a closed subgroup of a compact group \( G \) and \( T \) is an \( L^p(G) \)-convolution operator. Then \( T \) corresponds to an
$L^p(H)$-convolution operator $S$ by the rule $(Tf)|H=S(f|H)$, $f \in A(G)$, if and only if $\text{spt } T \subseteq H$. This correspondence is an isometry.

**Proof.** This follows immediately from the result of Herz [3, p. 317] that the restriction map $f \mapsto f|H$ from $A^p(G)$ to $A^p(H)$ is onto. □

**Remark.** For $G$ a locally compact abelian group and $H$ a closed subgroup, the analogous result of Corollary 4 has been shown by S. Saeki [5]. For $G$ a locally compact group and $H$ a compact normal subgroup, the analogous result to Corollary 4 has been shown by C. Herz ([3], [4]).

**Remark.** Let $G$ be a compact group and $H$ a normal closed subgroup. If $T \in M^p(G)$ such that $Tf=T(m_H * f)$, $f \in A(G)$, then there exists $S \in M^p(G/H)$ such that $Sf=Tf$ for $f \in A(G/H)=m_H * A(G)$ [1, p. 106], and $\|S\| = \|T\|$. Conversely, if $S \in M^p(G/H)$, then there exists $T \in M^p(G)$ defined by $Tf=S(m_H * f)$, $f \in A(G)$, and $\|T\| = \|S\|$. 

**Bibliography**


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