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DISCRETE SUBSETS OF PERFECTLY NORMAL SPACES

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ABSTRACT. Following R. W. Heath and E. A. Michael **[HM]**, we call a topological space perfect if each of its closed subsets is a G_δ.

In this note we (i) give a characterization of compact perfect T_1-spaces and (ii) obtain some results concerning the cardinality of discrete subsets of perfect spaces.

0. **Preliminaries.** We denote the cardinality of a set X by $|X|$, and we denote $2^{\aleph_0}$ by $\mathfrak{c}$.

As used here, the concepts of compactness, normality, etc. do not require that the T_1-separation axiom be satisfied. By a *countably compact* space we will mean a space in which each countable open cover has a finite subcover. A space will be called a *Lindelöf* space if each open cover has a countable subcover.

Several ideas due to the referee are used below, namely, the proof of Theorem 2, and the observation that the conclusion of Theorem 3 holds in a more general situation than in just a countably compact, noncompact one.

1. **A characterization of compact perfect T_1-spaces.** In **[GFW]**, S. L. Gulden, W. M. Fleischman, and J. H. Weston call a topological space $\aleph_0$-*bounded* if each of its countable subsets has compact closure.

DEFINITION. We shall call a space $\aleph_0$-*discrete-bounded* if each of its countable discrete subsets has compact closure.

For any topological space one can prove the following: compact $\Rightarrow$ $\aleph_0$-bounded $\Rightarrow$ $\aleph_0$-discrete-bounded $\Rightarrow$ countably compact. In a metric space all these properties are equivalent. For the larger class of perfect spaces, however, it is apparently not known if countable compactness implies compactness. In Theorem 1 we obtain a result in that direction.

THEOREM 1. *Let X be a T_1-space which is perfect and $\aleph_0$-discrete-bounded. Then X is compact.*

Theorem 1 is an immediate consequence of Theorems 2 and 3.

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THEOREM 2. *Let X be a T_1-space which is perfect and countably compact. Then every discrete subset Y of X is countable.*

PROOF. Let F be the set of all limit points of Y in X. Then F is closed in X, $Y \cap F = \varnothing$, and Y is closed in $X \setminus F$. Since $X \setminus F$ is open in X, $X \setminus F$ is an F_σ in X. Hence Y is also an F_σ in X. Because each closed discrete subspace of X is finite, Y must thus be countable.

THEOREM 3. *Let X be a space in which the closure of every countable discrete subset is a Lindelöf space, but suppose that X itself is not Lindelöf. Then X has an uncountable discrete subset.*

PROOF. Choose an open cover $\mathscr{U}$ of X which has no countable subcover, and let $\mathscr{M}$ be the family of all mappings f such that: (i) if $(x, V) \in f$ then $x \in X$, V is a neighborhood of x in X, and V is contained in some member of $\mathscr{U}$; and (ii) if (x, V), $(y, W) \in f$ with $x \neq y$, then $x \notin W$ and $y \notin V$. Let m be a maximal element of $(\mathscr{M}, \subset)$, and define $I=$domain of m and $\mathscr{D}=$range of m.

Clearly I is discrete, and so if it is uncountable, our proof is complete. Let us suppose that I is countable. Then $\bar{I}$ is Lindelöf, and hence $\bar{I} \subset \bigcup \mathscr{C}$ for some countable subcollection $\mathscr{C}$ of $\mathscr{U}$. By (i), $\bigcup \mathscr{D}$ is also covered by a countable subcollection of $\mathscr{U}$. Since no countable subcollection of $\mathscr{U}$ covers X, there must exist a point $x \in X$ such that $x \notin \bigcup \mathscr{C}$ and $x \notin \bigcup \mathscr{D}$. But if U is any set in $\mathscr{U}$ with $x \in U$ then $m \cup \{(x, U \cap (X \setminus \bar{I}))\}$ is a member of $\mathscr{M}$ strictly greater than m.

2. **The cardinality of discrete subsets of perfect spaces.** If a perfectly normal space X has a dense subset of cardinality $\mathfrak{M} \geqq \aleph_0$, then a simple counting argument shows that for every discrete subset I of X, one must have $2^{|I|} \leqq 2^{\mathfrak{M}}$. (A variety of examples show that if X is just perfect but not normal, then this inequality need not hold.)

In Theorem 2 we proved that if a perfect T_1-space satisfies a certain compactness condition, namely, countable compactness, then there is a sharper upper bound for the cardinalities of its discrete subsets. A related condition, one that is known to be implied by countable compactness, equivalent to it in normal spaces, and equivalent to pseudocompactness in completely regular spaces, is *feeble compactness*—the requirement that every locally finite system of open sets be finite. The example in [**GJ**, 5I] shows that in Theorem 2 countable compactness cannot be weakened to feeble compactness, even for a perfect, locally compact, zero-dimensional T_1-space; however, a proof similar to the proof of Theorem 2 shows that every *open* discrete subset of a perfect feebly compact space is countable.

It should be noted that, in the presence of the Continuum Hypothesis, Theorem 2 is known for the case in which the space X is compact

Hausdorff. For by a result of Alexandrov and Urysohn [**AU2**], every perfect compact Hausdorff space has no more than $\mathfrak{c}$ closed sets, and, obviously, no space with the latter property can have a discrete subset of cardinality $\geqq \mathfrak{c}$. We show next that the hypothesis of this theorem of Alexandrov and Urysohn (and also the hypotheses of the results established in [**AU1**, pp. 38–41]) can be weakened.

THEOREM 4. *Let X be a perfect T_1-space.* (i) *If X is either Lindelöf or countably compact then* $|X| \leqq \mathfrak{c}$. (ii) *If X is Lindelöf, and if each point $x \in X$ has a fundamental system of neighborhoods of cardinality $\leqq \mathfrak{c}$, then X has no more than $\mathfrak{c}$ closed sets.*

PROOF. (i) According to [**HJ**, Theorem 6], if X is a space in which (a) each point is a G_δ and (b) each discrete subset is countable, then $|X| \leqq \mathfrak{c}$. Since here X is T_1, perfect, and either countably compact or hereditarily Lindelöf, (a) and (b) are satisfied. (ii) Then X has a base $\mathscr{B}$ with $|\mathscr{B}| \leqq \mathfrak{c}$, and every closed subset of X can be obtained by taking a countable intersection of countable unions of members of $\mathscr{B}$.

The following space X is a perfect, feebly compact, extremally disconnected Hausdorff space of cardinality $> \mathfrak{c}$.

EXAMPLE 5. Let $N \subset Y \subset \beta N$ with Y pseudocompact and $|Y| > \mathfrak{c}$. Let X be the space whose points are those of Y, and which is topologized as follows: each point of N is isolated; a neighborhood of a point $p \in X \backslash N$ is any superset of a set of the form $\{p\} \cup (N \cap F)$ where F is a neighborhood of p in Y.

We conclude by raising two questions. (i) Is there a regular perfect T_1-space that is countably compact but not compact? (ii) Is there a compact Hausdorff space which has no more than $\mathfrak{c}$ closed sets but which is not perfect?

We observe that no subspace of a linearly ordered space can be used to obtain an affirmative answer to (i), for D. Lutzer [**L**] has proved that a perfect subspace of a linearly ordered space is hereditarily paracompact.

If $\mathfrak{c} = \exp(\aleph_1)$ then the ordered space consisting of all ordinals $\leqq \omega_1$ is a compact Hausforff space which is not perfect but in which there are no more than $\mathfrak{c}$ closed sets.

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