IRREDUCIBLE 3-MANIFOLDS WHOSE ORIENTABLE COVERS ARE NOT PRIME

W. H. ROW, JR.

Abstract. J. L. Tollefson has asked if every closed covering space of a prime 3-manifold is prime. In the present paper, the author provides a negative answer by constructing infinitely many topologically distinct, irreducible, closed 3-manifolds with the property that none of their orientable covering spaces are prime. These 3-manifolds are distinguished by the maximum number of disjoint, nonparallel, 2-sided projective planes that they contain. The author does not know if every closed covering space of a prime, orientable 3-manifold is prime.

1. Preliminaries. We will work in the piecewise linear category. Manifolds will be connected but not necessarily compact, orientable, or without boundary. Surfaces are compact 2-manifolds. A manifold \( N \) is properly embedded in a manifold \( M \) if \( N \subset M \) and \( N \cap \text{Bd } M = \text{Bd } N \). An \((n-1)\)-manifold \( N \), properly embedded in an \( n \)-manifold \( M \), is 2-sided if \( N \) separates some connected neighborhood of \( \text{Bd } N \) in \( M \).

A 3-manifold \( M \) is irreducible if every 2-sphere in \( M \) bounds a 3-cell in \( M \). A 3-manifold \( M \) is prime if every separating 2-sphere in \( M \) bounds a 3-cell in \( M \). A properly embedded surface \( S \) in a 3-manifold \( M \) is incompressible provided \( (1) \) if \( D \) is a 2-cell in \( M \) such that \( D \cap S = \text{Bd } D \), then \( \text{Bd } D \) bounds a 2-cell in \( S \), and \( (2) \) if \( S \) is a 2-sphere, then \( S \) does not bound a 3-cell in \( M \). The unit interval \([0, 1]\) will be denoted by \( I \).

If \( S_1, \cdots, S_m \) are disjoint, closed surfaces in a 3-manifold \( M \), we say \( N \) is a parallelity component of \( M - (S_1 \cup \cdots \cup S_m) \) if for some closed surface \( S \) and some \( S_i, S_j, i \neq j \), the closure of \( N \) is homeomorphic to \( S \times I \) and \( S_i, S_j \) correspond to \( S \times \{0\}, S \times \{1\} \), respectively. If \( M - (S_1 \cup S_2) \) has a parallelity component, then we say \( S_1 \) and \( S_2 \) are parallel. A consequence of Lemma 1 is that if \( S_1 \) and \( S_2 \) are parallel in \( M \), then \( S_1, \cdots, S_m \) have a parallelity component in \( M \).

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Lemma 1 is a special case of a more general result of Haken [1, pp. 91–96].

**Lemma 1.** If $S$ is a closed 2-manifold and $T$ is an incompressible surface in the interior of $S \times I$, then $T$ is parallel to $S \times \{0\}$.

For the following two lemmas let $T_1, \cdots, T_k$ be disjoint, 2-sided, incompressible, properly embedded surfaces in a 3-manifold $M$. We will follow Waldhausen [6, p. 57] in using $U(\cdots)$ for nice regular neighborhoods.

**Lemma 2.** $M$ is irreducible if and only if the closure of each component of $M - \bigcup U(T_i)$ is irreducible.

**Lemma 3.** If $P_1, \cdots, P_m$ are disjoint, 2-sided projective planes in $\text{Int } M$ and $M$ is irreducible, then there is a homeomorphism $h$ of $M$ onto itself, fixed on $\text{Bd } M$, such that $h(P_1), \cdots, h(P_m)$ do not intersect $T_1, \cdots, T_k$.

If $M$ is orientable and compact, Lemma 2 appears in [6, p. 59]. The same techniques prove Lemma 2. In Lemma 3, since $P_i$ and $T_j$ are 2-sided, any “general position” intersection curve $J$ of $P_i$ and $T_j$ is 2-sided on $P_i$. Hence $J$ bounds a 2-cell on $P_i$. Using this fact the techniques used for Lemma 2 also prove Lemma 3.

**Lemma 4.** Let $N$ be a closed 3-manifold that contains a 2-sided projective plane. Suppose $M$ is a prime, orientable cover of $N$. Then $M$ is homeomorphic to $S^2 \times S^1$.

**Proof.** Let $p$ be a covering map from $M$ to $N$. Suppose $P$ is a 2-sided projective plane in $N$. Each component of $p^{-1}(P)$ is a covering space for $P$ embedded in $M$ as a 2-sided subset and hence a 2-sphere. Let $S$ be one of the components of $p^{-1}(P)$. If $S$ separates $M$, then $S$ bounds a 3-cell $C$ in $M$. We may suppose $p^{-1}(P) \cap \text{Int } C$ is empty. Therefore $p(C)$ is a compact 3-manifold in $N$ with $\text{Bd } (p(C)) = P$, an impossibility. Since $M$ contains a nonseparating 2-sphere, Milnor [4, Lemma 1] has shown $M$ is homeomorphic to $S^2 \times S^1$.

2. Irreducible 3-manifolds that contain 2-sided projective planes. We will construct a 3-manifold $Q_k$ for each positive integer $k$. Subscripts will run from 1 to $k$ throughout this section. Let $a_1, \cdots, a_k$ denote distinct singleton subsets of the 1-sphere $S^1$. Suppose $J$ is a nonseparating simple closed curve in $P^2$, real projective 2-space. Then $K_j = U(J \times a_j) \subset P^2 \times S^1$ is a solid Klein bottle. Let $E_i$ be a properly embedded 2-cell in $K_i$ that intersects $J \times a_j$ in exactly one piercing point. Then $C_i = U(E_i)$ in $K_i$ and the closure of $K_i - C_i$ are both 3-cells. Note that $A_i = C_i \cap \text{Bd } K_i$ is an annulus. Then $N_k = P^2 \times S^1 - \bigcup \text{Int } K_i$ is a 3-manifold. Let $B_i$ be a 3-cell.
in $C_i - A_i$ such that $\text{Bd } B_i$ and $\text{Bd } C_i$ intersect in two 2-cells, one in each component of $\text{Bd } C_i - A_i$, and the closure of $C_i - B_i$ is a cube-with-a-knotted-hole $L_i$. Let $M_k = N_k \cup (\bigcup L_i)$. In the terminology of [3, pp. 379–380] we obtain $M_k$ from $N_k$ by attaching cubes-with-a-knotted-hole $L_i$ to $N_k$ along annuli $A_i$.

**Note.** $N_k$ and $M_k$ are irreducible, both being the closure of $P^2 \times S^1$ minus disjoint, solid Klein bottles. $A_i$ is not contractible in $M_k$, since otherwise, using the Loop Theorem, we could find a nonseparating 2-sphere in $P^2 \times S^1$. Hence $A_i$ is an incompressible, properly embedded surface in $M_k$.

Let $Q_k = 2M_k$, the double of $M_k$, obtained by sewing two copies of $M_k$ (denoted by $M_k$ and $M_k'$) together along their boundaries by the identity map. If $X$ is a subset of $M_k$, we will denote the corresponding subset of $M_k'$ by $X'$. We consider $M_k$ and $M_k'$ both to be subsets of $Q_k$.

Since the orientable double cover of $Q_k$ is closed, Theorem 5 answers a question of J. L. Tollefson [5, p. 106] in the negative.

**Remark.** Each $Q_k$ contains 2-sided projective planes. It may be of interest to note that if the Poincaré conjecture is true, Tollefson’s question has an affirmative answer for 3-manifolds that do not contain 2-sided projective planes.

**Theorem 5.** $Q_1, \ldots, Q_k, \ldots$ are topologically distinct, irreducible, closed 3-manifolds such that no orientable covering space of $Q_k$ is prime.

**Proof.** First we will show $Q_k$ is irreducible. In view of Lemma 2 and the Note, it is sufficient to show that each component of $\text{Bd } M_k$ is incompressible in $Q_k$. Suppose some component of $\text{Bd } M_k$ is compressible in $Q_k$. Then there is a 2-cell $D \subseteq Q_k$ such that $D \cap \text{Bd } M_k = \text{Bd } D$ and $\text{Bd } D$ does not bound a 2-cell in $\text{Bd } M_k$. (Note $\text{Bd } D$ is not necessarily contained in the component of $\text{Bd } M_k$ we assumed to be compressible.) We may assume that $D \subseteq M_k$. Returning to the notation we used in defining $M_k$, if $\text{Bd } D$ is contained in the component of $\text{Bd } M_k$ we assumed to be compressible.) We may assume that $D \subseteq M_k$. Returning to the notation we used in defining $M_k$, if $\text{Bd } D$ is contained in the component of $\text{Bd } M_k$ that intersects $L_i$, let $A, A^*$ denote the annuli obtained by taking the closures of $\text{Bd } L_i - A_i$ and $\text{Bd } K_i - A_i$, respectively. Then $\text{Bd } D \subseteq A \cup A^*$.

$\text{Bd } D$ must intersect $\text{Bd } A_i = \text{Bd } A = \text{Bd } A^*$. If not, then $\text{Bd } D$ is contained in $A$ or $A^*$. Since both $A$ and $A^*$ are not contractible in $M_k$, $\text{Bd } D$ bounds a 2-cell in $Bd M_k$.

After performing a small ambient isotopy we can assume $D$ and $A_i$ are in general position. The components of $D \cap A$ and $D \cap A^*$ are arcs. We will call a properly embedded arc in an annulus a spanning arc if the endpoints of the arc lie in different boundary components of the annulus. We can assume the components of $D \cap A$ and $D \cap A^*$ are spanning arcs, since nonspanning arcs can easily be removed by an ambient isotopy.
There exist an arc $Z$ in $D \cap A_i$ and 2-cells $F_1$ and $F_2$ such that $F_1 \cup F_2 = D$, $F_1 \cap F_2 = Z$, and $A_i \cap \partial D F_i = Z$. Then $Z$ is a spanning arc of $A_i$ since the closure of $\partial D F_1 - Z$ is a spanning arc of either $A$ or $A^*$.

If $\partial D F_1 \subset A_i \cup A^*$, then $\partial D F_1$ is a nontrivial simple closed curve on $\partial D L_i$. Since the inclusion map from $L_i$ into $M_k$ induces a monomorphism on fundamental groups, an application of the Loop Theorem shows $\partial D F_1$ must bound a 2-cell in $L_i$. But this contradicts the fact $L_i$ is a cube-with-a-knotted-hole.

If $\partial D F_1 \subset A_i \cup A^*$, then $\partial D F_1$ is a simple closed curve on $\partial D M_i$ that bounds an annulus with $J \times a_i$ in $K_i$. But then $J \times a_i$ is contractible in $P^2 \times S^1$. Hence $K_i$ is a solid torus, a contradiction. We have now shown that $Q_k$ is irreducible.

Let $P(M)$ denote the maximum number of disjoint, 2-sided, nonparallel projective planes that can be embedded in a compact 3-manifold; M. Haken ([1], [2]) has shown $P(M)$ exists. However we do not need Haken’s complete result for Lemma 6. $P(M)$ is clearly a topological property of $M$. Lemma 6 shows $Q_1, \ldots, Q_k, \ldots$ are distinct.

**Lemma 6.** $P(Q_k) = 2k$.

**Proof.** Let $S_1, \ldots, S_i$ be disjoint, 2-sided projective planes in $Q_k$. Returning to the notation used in defining $N_k$, if $K_i$ is properly chosen, $P^2 \times a_i$ intersects $N_k$ in a 2-cell $D_i$. Then the closures $R_1, \ldots, R_k$ of the $k$ components of $N_k - \bigcup (U(D_i))$ are each homeomorphic to $P^2 \times I$. Note that the components of $\partial D M_k$, the annuli $A_i$, the annuli $A_i'$, the 2-cells $D_i$, and the 2-cells $D_i'$ are incompressible, 2-sided, properly embedded surfaces in $Q_k, M_k, M_k^*, K_k, N^*$, respectively. By repeated applications of Lemma 3 we can find a homeomorphism $h$ of $Q_k$ onto itself such that $h(S_1), \ldots, h(S_i)$ miss the union of $\partial D M_k$ with the $A_i$, $A'_i$, $D_i$, and $D'_i$. Hence we may assume that $S_1, \ldots, S_i$ are contained in the 3-manifolds $R_1, \ldots, R_k, R'_i, \ldots, R'_k$. The cubes-with-a-knotted-hole $L_1, \ldots, L_k, L'_1, \ldots, L'_k$ cannot contain 2-sided projective planes. Lemma 1 implies that if $l > 2k$, then some $S_i$ and $S_j$ are parallel. So $P(Q_k) \leq 2k$.

Let $h_i$ be a homeomorphism from $P^2 \times I$ onto $R_i$. Let $P_i = h_i(P^2 \times \{\frac{1}{2}\})$. Then $P_1, \ldots, P_k, P'_1, \ldots, P'_k$ are disjoint, 2-sided projective planes in $Q_k$. Since $\bigcup P_i$ and $\bigcup P'_i$ each fail to separate $Q_k$, only a $P_i$ and a $P'_j$ could bound a parallelity component $N$. But $\partial D M_k$ separates $P_i$ and $P'_j$ in $Q_k$. So $N$ contains a component of $\partial D M_k$, an incompressible surface of Euler characteristic zero. This contradicts Lemma 1. Hence $P(Q_k) \geq 2k$.

All that remains to prove Theorem 5 is to show no orientable covering space of $Q_k$ is prime. By Lemma 4 the only orientable, prime cover $Q_k$ could have is $S^2 \times S^1$. Let $p$ be a covering map from $S^2 \times S^1$ to $Q_k$. Now $p^{-1}(\partial D M_k)$ is a collection of incompressible, closed surfaces of Euler...
characteristic zero. But the only incompressible surfaces $S^2 \times S^1$ contains are 2-spheres. Hence $S^2 \times S^1$ cannot be a cover of $Q_k$. The proof is complete.

REFERENCES

2. ———, *Erratum for some results on surfaces in 3-manifolds* (mimeographed notes).