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GEÖCZE AREA AND A CONVERGENCE PROPERTY

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ABSTRACT. Suppose f is a continuous mapping with finite Lebesgue area from a polyhedral region $X \subset R^k$ into R^n, $2 \leqq k \leqq n$. Let $f = l \circ m$ be the monotone-light factorization of f with middle space M.

If f satisfies a "cylindrical condition" considered by T. Nishiura, then a current valued measure T over M can be associated with f by means of the Cesari-Weierstrass integral, and if $\{f_i\}$ is any sequence of quasi-linear maps $f_i: X \to R^n$ converging uniformly to f with bounded areas, then

$$T(g)(\phi) = \lim_{i\to\infty} \int_X (g \circ m) f_i^{\#} \phi$$

whenever ϕ is an infinitely differentiable k-form in R^n and g is a continuous real valued function on M which vanishes on $m(\text{Bdry } X)$.

The total variation measure of T, taken with respect to mass, coincides with the Geöcze area measure over M.

1. Suppose f is a continuous mapping with finite Lebesgue area from a polyhedral region $X \subset R^k$ into R^n, $2 \leqq k \leqq n$. Let $f = l \circ m$ denote the monotone-light factorization of f with middle space M.

Suppose $p: R^n \to R^k$ is an orthogonal projection. Let $p \circ f = \bar{l} \circ \bar{m}$ denote the monotone-light factorization of $p \circ f$ with middle space $\bar{M}$ and let h be the monotone map such that the diagram

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[dr, "\bar{m}"'] & M \arrow[r, "l"] \arrow[d, "h"] & R^n \arrow[d, "p"] \\ & \bar{M} \arrow[r, "\bar{l}"'] & R^k \end{tikzcd}$$

commutes.

Let $C_p = M \cap \{z : \text{diam } h^{-1}(h(z)) > 0\}$, where diameter is taken with respect to the usual metric in M. It was shown in [N1, 2.1] that, if either $k=2$ or the $k+1$ dimensional Hausdorff measure of $f(X)$ is zero,

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then the k dimensional Lebesgue measure of $p \circ l(C_p)$ is zero for every orthogonal projection p.

Let $\Lambda(k, n)$ denote the set of all k-tuples $\lambda=(\lambda_1, \cdots, \lambda_k)$ of integers such that $1 \leqq \lambda_1 < \cdots < \lambda_k \leqq n$ and let $p^\lambda : R^n \to R^k$ be the orthogonal projection defined by

$$p^\lambda(y) = (y_{\lambda_1}, \cdots, y_{\lambda k}) \quad \text{for } y = (y_1, \cdots, y_n) \in R^n.$$

Let $C(M)$ denote the space of all continuous real valued functions on M and let $C_0(M)$ denote the space of those $g \in C(M)$ which vanish on $m(\text{Bdry } X)$.

Let $e_1, \cdots, e_n$ be the standard basis in R^n and let $e_\lambda = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_k}$, $\lambda \in \Lambda(k, n)$, be the corresponding basis for the space of k-vectors in R^n.

For $\lambda \in \Lambda(k, n)$ let $f^\lambda = p^\lambda \circ f$ and for each simple polyhedral region $\pi \subset X$ let

$$u(f^\lambda, \pi) = \int_{R^k} O(f^\lambda, \pi, y)\, dy$$

where $O(f^\lambda, \pi, y)$ denotes the topological index of $y \in R^k$ with respect to the mapping $f^\lambda | \pi$ if $y \in R^k \backslash f^\lambda(\text{Bdry } \pi)$ and $O(f^\lambda, \pi, y) = 0$ if $y \in f^\lambda(\text{Bdry } \pi)$. Let

$$u(f, \pi) = \sum_{\lambda \in \Lambda(k,n)} u(f^\lambda, \pi) e_\lambda.$$

For any finite nonoverlapping collection P of simple polyhedral regions $\pi \subset X$ let

$$\begin{aligned} \delta(P) = {} & \max\{\operatorname{diam} f(\pi) : \pi \in P\} \\ & + \max\left\{ V(f^\lambda) - \sum_{\pi \in P} |u(f^\lambda, \pi)| : \lambda \in \Lambda(k, n) \right\} \end{aligned}$$

where $V(f^\lambda)$ is the Geöcze area of f^λ.

In [**G**, Theorem 1] it was shown that if the infimum of the numbers $\delta(P)$, taken over all P as above, is zero, then the Cesari-Weierstrass integral

$$T(g)(\phi) = \lim_{\delta(P) \to 0} \sum_{\pi \in P} \frac{1}{|\pi|} \int_\pi g \circ m(x) \phi(f(x)) \cdot u(f, \pi)\, dx$$

exists whenever $g \in C(M)$ and ϕ is an infinitely differentiable k-form on R^n. Here $|\pi|$ denotes the k dimensional Lebesgue measure of π. The current valued linear mapping T on $C(M)$ possesses a unique extension to the class of all bounded Borel measurable functions on M such that Lebesgue's bounded convergence theorem holds.

THEOREM. *If the k dimensional Lebesgue measure of* $p^\lambda \circ l(C_p\lambda)$ *is zero for each* $\lambda \in \Lambda(k, n)$, *then*:

(1) *A current valued measure T can be associated with f by means of the Cesari-Weierstrass integral.*

(2) *If* $\{f_i\}$ *is any sequence of quasi-linear mappings* $f_i : X \to R^n$ *converging uniformly to f with bounded areas, then*

$$T(g)(\phi) = \lim_{i \to \infty} \int_X (g \circ m) f_i^{\#} \phi$$

whenever $g \in C_0(M)$ *and* $\phi \in E^k(R^n)$.

(3) *The total variation measure* $\|T\|$ *of T, taken with respect to mass, coincides with the Geöcze area measure over M.*

The notation is that of [**F1**] and [**G**].

In case either $k=2$ or the $k+1$ dimensional Hausdorff measure of $f(X)$ is zero, the conclusions of the Theorem above can be obtained using the results of [**F1**] and [**G**]. It is the purpose of this note to show that the given weaker hypothesis suffices; in particular that one can avoid the use of [**G**, Theorem 5].

Conclusion (2) of the Theorem should be compared with [**F1**, 3.10]. Note that $f|X \backslash m^{-1}(m(\mathrm{Bdry}\, X))$ need not be proper

2. **The case** $k = n$. Suppose f is a continuous mapping with finite Lebesgue area from a polyhedral region $X \subset R^k$ into R^k, $2 \leq k$. By [**G**, Theorems 1 and 2] applied with $k=n$, the current valued measure T associated with f by means of the Cesari-Weierstrass integral exists and the total variation measure $\|T\|$ of T coincides with the Geöcze area measure over M.

The arguments of [**F1**, 3.4 and 3.6] show that there exists a unique current valued measure S over $M \backslash m(\mathrm{Bdry}\, X)$ such that, for every sequence $\{f_i\}$ of quasi-linear maps $f_i : X \to R^k$ converging uniformly to f with bounded areas, we have

$$S(g)(\phi) = \lim_{i \to \infty} \int_X (g \circ m) f_i^{\#} \phi$$

for $g \in C_0(M)$ and $\phi \in E^k(R^k)$.

Since the Lebesgue area of f and the Geöcze area of f are equal when $k=n$, there exists a sequence $\{f_i\}$ of quasi-linear maps converging uniformly to f with areas converging to the Geöcze area of f. Application of [**G**, Theorem 4] to this case shows that

$$T(g)(\phi) = \lim_{i \to \infty} \int_X (g \circ m) f_i^{\#} \phi$$

for $g \in C_0(M)$ and $\phi \in E^k(R^k)$.

3. **Proof of the Theorem.** Suppose $\{f_i\}$ is any sequence of quasi-linear maps converging uniformly to f with bounded areas. As in [**F1**, 3.4] there is a subsequence, which we continue to denote by $\{f_i\}$, such that

$$S(g)(\phi) = \lim_{i\to\infty} \int_X (g \circ m) f_i^{\#} \phi$$

exists whenever $g \in C(M)$ and $\phi \in E^k(R^n)$. The current valued mapping S so defined possesses a unique extension, which we continue to denote by S, to the class of all bounded Borel measurable functions on M such that Lebesgue's bounded convergence theorem holds. With obvious modifications, the argument of [**F1**, 3.4] shows that $S(B)$ is a rectifiable current whenever B is a Borel subset of $M\backslash m(\text{Bdry } X)$.

From [**N1**] we see that the infimum of the numbers $\delta(P)$, taken over all finite collections P of nonoverlapping simple polyhedral regions $\pi \subset X$, is zero and hence we can define [**G**, Theorem 1] a current valued function T on $C(M)$ by letting

$$T(g)(\phi) = \lim_{\delta(P)\to 0} \sum_{\pi \in P} \frac{1}{|\pi|} \int_\pi g \circ m(x)\phi(f(x)) \cdot u(f, \pi)\, dx$$

whenever $g \in C(M)$ and $\phi \in E^k(R^n)$. We denote by T also the unique extension of this current valued mapping to the class of all bounded Borel measurable functions on M such that Lebesgue's bounded convergence theorem holds.

Let $e^1, \cdots, e^n$ denote the usual basis for the space of covectors in R^n and let $e^\lambda = e^{\lambda_1} \wedge \cdots \wedge e^{\lambda_k}$, $\lambda \in \Lambda(k, n)$, denote the corresponding basis for the space of k-covectors in R^n.

Let $u \in E^0(R^n)$ be such that $u(x) = 1$ for $x \in R^n$ and, for $\lambda \in \Lambda(k, n)$, consider the signed Borel measures defined over M by letting

$$S^\lambda(B) = S(B) \wedge e^\lambda(u) \quad \text{and} \quad T^\lambda(B) = T(B) \wedge e^\lambda(u)$$

whenever B is a Borel subset of M.

In order to prove statement (2) it suffices to show that $S^\lambda(B) = T^\lambda(B)$ whenever B is a Borel subset of $M\backslash m(\text{Bdry } X)$.

For fixed $\lambda \in \Lambda(k, n)$ let $l_\lambda \circ m_\lambda$ denote the monotone-light factorization of $f^\lambda = p^\lambda \circ f$ with middle space M_λ and let h be the monotone map such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "m"] \arrow[dr, "m_\lambda"'] & M \arrow[r, "l"] \arrow[d, "h"] & R^n \arrow[d, "p^\lambda"] \\
& M_\lambda \arrow[r, "l_\lambda"'] & R^k
\end{tikzcd}$$

From the case $k=n$, previously considered, we have $S^\lambda(\psi \circ h)=T^\lambda(\psi \circ h)$ whenever $\psi \in C_0(M_\lambda)$. By Lebesgue's bounded convergence theorem the above equality holds for any bounded Borel measurable function ψ on M_λ that vanishes on $m_\lambda(\text{Bdry } X)$. In particular

$$S^\lambda(h^{-1}(B)) = T^\lambda(h^{-1}(B))$$

whenever B is a Borel subset of $M_\lambda \backslash m_\lambda(\text{Bdry } X)$.

For any Borel set $A \subset M\backslash[C_{p^\lambda} \cup m(\text{Bdry } X)]$, we have

$$h(A) \subset M_\lambda\backslash m_\lambda(\text{Bdry } X) \quad \text{and} \quad h^{-1}(h(A)) = A.$$

Hence $S^\lambda(A)=T^\lambda(A)$.

We will complete the proof of statement (2) by showing that $T^\lambda(B)$ and $S^\lambda(B)$ both vanish for any Borel set $B \subset C_{p^\lambda}\backslash m(\text{Bdry } X)$.

Let μ_λ denote the finite Borel measure [**N2**, 6.10] defined over M_λ by letting

$$\mu_\lambda(B) = \inf\{V(f^\lambda \mid m_\lambda^{-1}(U)) : U \text{ open in } M_\lambda,\ U \supset B\}$$

whenever B is a Borel subset of M_λ. Here $V(f^\lambda | m_\lambda^{-1}(U))$ denotes the Geöcze area of $f^\lambda | m_\lambda^{-1}(U)$.

For $\eta>0$ let $C^\eta = M \cap \{z : \text{diam } h^{-1}(h(z)) \geqq \eta\}$. Then each C^η is closed and $C_{p^\lambda} = \bigcup_{\eta>0} C^\eta$.

Suppose $\eta>0$ and P is a finite collection of nonoverlapping simple polyhedral regions $\pi \subset X$ with $\delta(P)<\eta$. Then, for each $\pi \in P$, we have

$$m_\lambda(\pi) \cap h(C^\eta) \subset m_\lambda(\text{Bdry } \pi).$$

By [**N2**, 6.5], $V(f^\lambda|\pi)=\mu_\lambda(m_\lambda(\pi)-m_\lambda(\text{Bdry } \pi))$ and hence

$$\begin{aligned} \mu(M_\lambda) = V(f^\lambda) &\leqq \sum_{\pi \in P} V(f^\lambda \mid \pi) + \delta(P) \\ &= \sum_{\pi \in P} \mu_\lambda(m_\lambda(\pi) - m_\lambda(\text{Bdry } \pi)) + \delta(P) \\ &\leqq \mu_\lambda(M_\lambda \backslash h(C^\eta)) + \delta(P). \end{aligned}$$

Thus $\mu_\lambda(h(C^\eta))=0$ for each $\eta>0$ and hence $\mu_\lambda(h(C_{p^\lambda}))=0$.

If U is open in M_λ, then $h^{-1}(U)$ is open in M and $m_\lambda^{-1}(U)=m^{-1}(h^{-1}(U))$. Thus

$$\inf\{V(f^\lambda \mid m^{-1}(U)) : U \text{ open in } M,\ U \supset C_{p^\lambda}\} = 0.$$

(This argument was suggested by T. Nishiura [**N3**, 4.5].) Since

$$T^\lambda(\psi) = \lim_{\delta(P)\to 0} \sum_{\pi \in P} \frac{1}{|\pi|} \int_\pi \psi \circ m(x) u(f^\lambda, \pi)\, dx$$

for $\psi \in C(M)$, we infer that $T^\lambda(B)=0$ whenever B is a Borel subset of C_{p^λ}.

If B is a Borel subset of $C_{p\lambda}\backslash m(\text{Bdry } X)$ whose closure is contained in $C_{p\lambda}$, then $S(B)$ is rectifiable and spt $S(B) \subset l(C_{p\lambda})$. Thus, by **[F2**, 2.1], $S^{\lambda}(B)=0$. Since $C_{p\lambda}$ is an F_{σ} set, we have $S^{\lambda}(B)=0$ for each Borel set $B \subset C_{p\lambda}\backslash m(\text{Bdry } X)$ and statement (2) follows.

Statement (3) now follows from **[G**, Theorems 3 and 7].

4. **Example.** We consider here an example from **[N1**, 2.2] to show that, in case $2<k<n$, the hypothesis of the Theorem may be satisfied even though the $k+1$ dimensional measure of $f(X)$ is positive.

Let X_0 denote the unit square in R^2 and let $\bar{f}$ denote a continuous mapping with finite area from X_0 into R^3 such that $\bar{f}(X_0)$ has positive 3 dimensional measure. If $\bar{p}: R^3 \to R^2$ is any orthogonal projection and $\bar{l}$ is the light factor of $\bar{f}$, then the 2 dimensional measure of $\bar{p} \circ \bar{l}(C_{\bar{p}})$ is zero.

Let $X=X_0 \times I$ where I is the unit interval and let f denote the continuous mapping with finite Lebesgue area from X into R^4 defined by letting

$$f(x) = (\bar{f}(x_1, x_2), x_3) \quad \text{for } x = (x_1, x_2, x_3) \in X.$$

The 4 dimensional measure of $f(X)$ is positive. If $p: R^4 \to R^3$ is an orthogonal projection such that $p(e_4)=0$ and l is the light factor of f, then the 3 dimensional measure of $p \circ l(C_p)$ is equal to that of $\bar{f}(X_0)$ and hence is positive. If, however, $p(e_4) \neq 0$, then it is readily seen that the 3 dimensional measure of $p \circ l(C_p)$ is zero.

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