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SOME THEOREMS ON FREDHOLM MAPS

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ABSTRACT. Let $f: M \to N$ be a C^1 proper Fredholm map of index n of C^r Banach manifolds. Then under certain hypotheses one may approximate f by a C^r proper Fredholm map g of index n with g C^1 properly homotopic to f.

The rank theorem for Fredholm maps is stated and it is then used to obtain the invariance of domain for this class of maps and also a nonlinear version of the Fredholm alternative theorem.

In recent years nonlinear Fredholm maps introduced by Smale [9] have been studied extensively by several authors [1], [2], [5], [6] and applications have been given to nonlinear elliptic boundary value problems [3]. One of the basic tools in the differential theory of such maps is that of approximating a C^1-Fredholm by one of class C^r, $r>1$. In the first section below we show that in certain cases we may approximate a C^1 proper Fredholm map f by a C^r proper Fredholm map g of the same index with g C^1 properly homotopic to f.

In §B we state the rank theorem for Fredholm maps and use it to obtain a nonlinear version of the Fredholm alternative theorem and an invariance of domain theorem.

The author is indebted to D. K. Elworthy whose ideas inspired this paper.

A. We begin with a few definitions and basic results. Let E and F be Banach spaces. A linear map $T \in L(E, F)$ is Fredholm if

(i) dim Ker $T < \infty$,

(ii) dim Coker $T < \infty$.

If M and N are Banach manifolds, a C^1 map $f: M \to N$ is said to be Fredholm if $df_p: TM_p \to TN_{f(p)}$ is linear Fredholm for each $p \in M$. If M is connected we may define the index of f to be

$$\operatorname{ind}(f) = \dim \operatorname{Ker} df_p - \dim \operatorname{Coker} df_p.$$

This does not depend on the choice of p.

A point $y \in N$ is said to be a regular value of f if whenever $x \in f^{-1}(y)$, $df_x: TM_x \to TN_y$ is surjective. If y is not a regular value it is a critical value.

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The following generalization of the Sard theorem due to Smale is the cornerstone of the modern differential theory of Fredholm maps.

THEOREM 1 (SARD-SMALE). *Let M be a second countable Banach manifold with $f: M \to N$ a C^3 Fredholm map of index n. Then if $q > \max(n, 0)$ the regular values are of second category in N (or the critical values are of first category). If f is proper (i.e., K compact $\Rightarrow f^{-1}(K)$ compact) then the regular values are open and dense.*

Let M be a C^1 Finsler manifold [7]. Then M admits a metric ρ (the Finsler metric). A set $X \subset M$ is *bounded* if, with respect to the metric ρ, $\operatorname{diam}(X) = \sup_{x,y \in X} \rho(x, y) < \infty$.

Suppose N is another Finsler manifold with a Finsler metric ρ'. Given maps $f: M \to N$ and $\delta: M \to R_+$, a map $g: M \to N$ is said to be a δ-approximation to f if $\rho'(f(x), g(x)) \leqq \delta(x)$ for all $x \in M$.

The topology on $C(M, N)$, the continuous maps from M to N generated by neighborhoods of the form

$$W(f, \delta) = \{g \in C(M, N) \mid \rho'(f(x), g(x)) < \delta(x)\}$$

where $\delta: M \to R_+$ is continuous, is called the C^0-fine topology on $C(M, N)$. It is not difficult to show that C^0 approximation does not depend on the metric ρ'.

Now again let M, N be Finsler manifolds with metrics ρ, ρ'. A map $h: M \to N$ is called *compact* if the image of ρ-bounded sets is relatively compact in N.

DEFINITION. Suppose $f_0: M_0 \to E_0$ is a C^1 Fredholm map of index n of the C^r Finsler manifold M_0, modelled on F_0 with metric ρ_0, to the Banach space E_0. A *smoothing space* of class C^r for the triple (f_0, M_0, E_0) is a triple (f_1, M_1, E_1) where M_1 is a C^r Finsler manifold modelled on F_1 admitting C^r partitions of unity with metric ρ_1, $\rho_1 \leqq k\rho_0$, k some positive constant. In addition we assume $M_0 \subset M_1$, $E_0 \subset E_1$, $F_0 \subset F_1$ are dense inclusions with the inclusion map $i: M_0 \to M_1$ compact. We assume also the f_1 is Fredholm of index n making the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & E_1 \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \mathrm{inc}} \\ M_0 & \xrightarrow{f_0} & E_0 \end{array}$$

commutative. Lastly we suppose that there is a differential structure on M_1 *uniform with respect to* M_0; that is a maximal atlas of charts on M_1, $(V_\beta, \psi_\beta)_{\beta \in B}$ with $\psi_\beta | V_\beta \cap M_0$ a C^1 chart for M_0.

The smoothing space is second countable if M_1 is. To see how smoothing spaces arise in the study of nonlinear elliptic boundary value problems see [**3**].

The proof of the following theorem is a minor modification of the argument in [**3**].

THEOREM 2. *Let $f: M_0 \to E_0$ be Fredholm of index n. If there exists a C^1 smoothing space for the triple (f_0, M_0, E_0) then f_0 is a proper map when restricted to bounded subsets of M_0.*

The proof of this theorem involves the notion of a C-structure on a Banach manifold which we now define. Let $\mathrm{GL}_c(E) \subset \mathrm{GL}(E)$ be the subgroup of the general linear group of E consisting of elements of the form $I+K$, K completely continuous. A *C-structure* on a manifold M is a maximal atlas of charts $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ covering M so that when defined $D(\varphi_\alpha \circ \varphi_\beta^{-1})\varphi_\beta(x) \in \mathrm{GL}_c(E)$.

A linear operator $T \in L(E \times R^n, E)$ is a $C(\pi_n)$ linear map if $T = \pi_n + K$, where $\pi_n: E \times R^n \to E$ is the natural projection on the first factor and K is completely continuous (here we take $\pi_0 = \mathrm{id}_E$). Suppose M is modelled on $E \times R^n$ as a C-structure M_c. A Fredholm map f of index n is a $C(\pi_n)$ map if, with respect to charts $(\varphi, V) \in M_c$, $D(f \circ \varphi^{-1})_x$ is $C(\pi_n)$.

The following basic result concerning $C(\pi_n)$ maps is proved in [**2**].

THEOREM 3. *Let M be modelled on F, $f: M \to E$ Fredholm of index n. Then there exists a unique C-structure $\{M, f\}_c$ on M modelled on $E \times R^n$ with respect to which f is a $C(\pi_n)$ map.*

REMARK. If (f_1, M_1, E_1) is a smoothing space for (f_0, M_0, E_0) the induced C-structure on M_1 can be made uniform with respect to M_0.

Before getting to the main result we will need a sequence of three lemmas.

LEMMA 1. *Let $U \subset E \times R^n$ be an open subset where E admits C^r, $r > 1$, partitions of unity. Suppose $f: U \to E$* is *a C^1 $C(\pi_n)$ map. Then given any $\delta > 0$ there exists a C^r $C(\pi_n)$ map g with $\|g(x) - f(x)\| < \delta$. Moreover, if $E_0 \subset E$ is dense, g can be chosen so that $g(x) - \pi_n(x) \in E_0$.*

PROOF. Let $V_y(\delta/2)$ be a covering of E by balls of radius $\delta/2$ about the point y (we may choose $y \in E_0$), and let $\{W_y\}_{y \in E}$ be an open cover of U defined by $W_y = \hat{f}^{-1}(V_y)$ where $\hat{f}(x) = f(x) - \pi_n(x)$.

Let $\{\hat{W}_j\}_{j \in J}$ be a locally finite open refinement of $\{W_y\}_{y \in F}$ covering U with $\{\varphi_j\}$ a C^r partition of unity subordinate to the covering $\{\hat{W}_j\}$. Choose V_{y_j} so that $\hat{f}(\hat{W}_j) \subset V_{y_j}$. Define

$$\hat{g}(x) = \sum \varphi_j(x) y_j, \qquad g(x) = \pi_n(x) + \hat{g}(x).$$

Then

$$\|f(x) - g(x)\| = \|\hat{f}(x) - \hat{g}(x)\| = \left\| \sum \varphi_j(x) \{\hat{f}(x) - y_j\} \right\|.$$

If $\|\hat{f}(x)-y_j\|>\delta/2$ then $\varphi_j(x)=0$. Thus $\|\hat{f}(x)-\hat{g}(x)\|<\delta$. Moreover since $x\mapsto\sum\varphi_j$ is locally finite dimensional, g is a C^r $C(\pi_n)$ map and satisfies the requirements of the lemma.

LEMMA 2. *Suppose $\delta: U\to R_+$ is continuous. Then the above lemma is true for nonconstant δ.*

PROOF. For each x, let

$$W_x = \{z\in U \mid |\delta(z)-\delta(x)| < \delta(x)/2\}.$$

Let $\{W_j\}_{j\in J}$ be a locally finite open refinement of $\{W_x\}_{x\in U}$ covering U, and let x_j be chosen so that $W_j\subset W_{x_j}$. Let $f_j: W_j\to F$ be the restriction of f to W_j and $\{\mu_j\}_{j\in J}$ be a C^r partition of unity subordinate to W_j. By the last lemma there is a C^r $C(\pi_n)$ map with $\|g_i(x)-f_i(x)\|<\delta(x_i)/2$. Then $g(x)=\sum\mu_j(x)g_j(x)$ will be a $C(\pi_n)$ map of U onto E and

$$\|g(x)-f(x)\| \leqq \sum\varphi_j(x)\,\|g_j(x)-f(x)\| \leqq \sum\varphi_j(x)(\delta(x_j)/2).$$

If $x\in W_j$, $\delta(x_j)/2<\delta(x)$ which implies that $\|g(x)-f(x)\|<\delta(x)$.

LEMMA 3. *Let W, V, U be open sets in $E\times R^n$ with $\overline{W}\subset V\subset\overline{V}\subset U$ and $f: U\to E$ a C^1 $C(\pi_n)$ map and $\delta: U\to R_+$ continuous. Then there exists a C^r $C(\pi_n)$ map $g: U\to E$ with $g|W$ a C^r $C(\pi_n)$ map $g|(U-\overline{V})=f|(U-\overline{V})$ and $\|f(x)-g(x)\|<\delta(x)$. Moreover there is a C^1 homotopy f_t of f so that $f_0=f$, $f_1=g$ with $f_t|(U-\overline{V})=f|(U-V)$ and $\|f_t(x)-f(x)\|<\delta(x)$ for all $t\in[0,1]$.*

PROOF. Consider $\hat{f}=f|V$. By Lemma 2 there is a C^r δ-approximation $\hat{g}: V\to F$ to $\hat{f}$ which is a $C(\pi_n)$ map. Let $\{\mu_i\}_{i=1,2}$ be a partition of unity subordinate to the covering V, $U-\overline{W}$ of U with support of μ_i contained in V. Define $\hat{g}(x)=\mu_1(x)\hat{g}(x)+\mu_2(x)f(x)$ and $f_t=(1-t)g+tf$.

REMARK. A slight modification of Lemma 3 shows that if f is a C^r $C(\pi_n)$ map on a neighborhood of a closed set $A\subset U$ then we can choose g so that $g|A=f|A$, and the same for f_t.

THEOREM 4. *Let $f_0: M_0\to E_0$ be a Fredholm map of index n with C^r smoothing space (f_1, M_1, E_1). Then given $\delta: M_1\to R_+$ there exists a C^r Fredholm map $g_0: M_0\to E_0$ of index n which is a δ-approximation to f_0 and admits a smoothing space (g_1, M_1, E_1). Moreover there exists a homotopy $F_0: M_0\times I\to E_0$ between f and g which has a C^r smoothing space $\{F_1, M_1\times I, E_1\}$, F_1^t a homotopy between g_1 and f_1 which is a δ-approximation to f for each $t\in[0,1]$.*

PROOF. We first assume that M_1 is second countable, thus all index sets will be countable. By Theorem 3, M_1 admits a C-structure $\{M_1, f_1\}_c$ with respect to which f_1 is a $C(\pi_n)$ map and the C-structure is uniform with respect to M_0.

Let U_i be an open cover of M_1 with U_i bounded and $(U_\alpha, \varphi_\alpha)$ a coordinate system in the C-structure $\{M_1, f_1\}_c$. Let $\{G_\alpha\}$, $\{V_\alpha\}$ be locally finite open coverings of M_1 with $\bar{G}_\alpha \subset V_\alpha$, $\bar{V}_\alpha \subset U_\alpha$, $\operatorname{dist}(\bar{G}_\alpha, V_\alpha^c) > 0$. Let $\eta < \delta$ and $g_0 = f$. Lemma 3 gives us a map $g_1^1 : M_1 \to E_1$ which is a $C(\pi_n)$ map on W_1 where $\bar{G}_1 \subset W_1$, $\bar{W}_1 \subset V_1$, $\operatorname{dist}[\bar{G}_1, W_1^c] > 0$, and g_1^1 is equal to f on the complement of V_1. Moreover we get that g_1^1 is C^1 homotopic to f by just a linear homotopy, and it is clear that g_1^1 restricts to M_0 to give an approximation g_0^1 to f_0.

Suppose we have defined a $C(\pi_n)$ map $g_1^{s-1} : M_1 \to E_1$ which restricts to a $C(\pi_n)$ map $g_0^{s-1} : M_0 \to E_0$ where g_1^{s-1} is C^r on a neighborhood W_{s-1} of $\bigcup_{k=1}^{s-1} \bar{G}_k$ which is a $(1-1/2^{s-1})\eta$ approximation to f. Let $A_s = \bigcup_{k=1}^{s} \bar{G}_k$ and take open neighborhoods H of A_{s-1} and B of $\bar{G}_s$ with $\bar{H} \subset W_{s-1}$ and $\bar{B} \subset V_s$. By the remark after Lemma 3 there is a map $\bar{g}_1^s : U_s \to E_1$ which is a C^r $C(\pi_n)$ map on B and equal to g_1^{s-1} outside V_s and on $V_s \cap \bar{H}$ and also on $\eta/2^s$ approximation to g_1^{s-1}. This extends over M_1 to give a map $g_1^s : M_1 \to E_1$ which is C^r $C(\pi_n)$ in a neighborhood of A_s restricts to g_1^{s-1} on A_{s-1}, is a $(1-1/2^s)\eta$ approximation to f and is homotopic to g_1^{s-1} by a homotopy which is constant on A_{s-1}. Moreover all C^r maps restrict naturally by construction to C^r maps from M_0 to E_0.

Thus we can define the map $g_1 : M_1 \to E_1$ and hence $g_0 : M_0 \to E_0$ inductively. If M_1 is not second countable we obtain the result using the same method but with refinements of the type considered in Lemma 2.4 of **[8]**.

Before proving the main theorem of this section we need a definition.

DEFINITION. Let (M, ρ), (N, ρ') be Finsler manifolds with metrics ρ and ρ' respectively. A map $f : M \to N$ is B-proper if:

(i) if $X \subset M$ is bounded, $f|X$ is proper;

(ii) if $Y \subset N$ is bounded then $f^{-1}(Y)$ is bounded in X.

It is clear that B-proper maps are proper. In addition in applications to nonlinear elliptic boundary value problems it is the B-properness that one usually verifies.

THEOREM 5. *Let $f_0 : M_0 \to E_0$ be a C^1 B-proper Fredholm map of index n, with C^r smoothing space (f_1, M_1, E_1). Then given $\delta : M_1 \to R_+$ there is a C^r δ-approximation g_0 to f_0 which is B-proper Fredholm of index n. Moreover there is a B-proper map $F_0 : M_0 \times I \to E_0$ so that for each $t \in I$, F_0^t is a Fredholm map of index n, and a δ-approximation to f_0.*

PROOF. Let $\eta < \delta < 1$. The last theorem gives an η-approximation g to f with C^r smoothing space (g_1, M_1, E_1), and also a homotopy F_0^t of f_0 and g_0. By Theorem 2, g_0 is proper on bounded sets. Also $\{x \mid \|g_0(x)\| \leqq C\} \subseteq \{x \mid \|f_0(x)\| \leqq C+1\}$ which shows that g_0 is B-proper if f_0 is. The B-properness of F_0 follows similarly.

REMARK. These results easily extend to the case where E_0 is replaced by a Finsler manifold N_0 with a *fixed* C-structure.

B. Let $f: M \to N$ be a proper $\mathscr{F}_0$ (Fredholm of index zero) map between two Banach manifolds. In [**9**] Smale showed that the cardinality of $f^{-1}(y)$ mod 2, for y a regular value, was independent of the point y. The mod 2 degree of f, denoted by $\gamma(f)$, has the property that if $\gamma(f) \neq 0$ then f must be surjective. In this section we will use this fact and an infinite dimensional version of the rank theorem to obtain a nonlinear extension of one of the standard Fredholm alternative theorems (see Widom [**10**]).

THEOREM 1 (RANK THEOREM FOR FREDHOLM MAPPINGS). *Let $b \in A \subset E$, A open, $f: A \to F$, a Fredholm mapping of index p with* dim Ker $Df_x = s$ *for $x \in A$. Then there exists a Banach space G, neighborhoods U of b and $V \supset f(U)$ of $f(b)$ and diffeomorphisms $\varphi: B_s \to U$, $\psi: V \to B_{s-p}$, where B_k is the open unit ball in $G \times E^k$, such that*

$$\psi \circ f \circ \varphi = \mathrm{id}_G \times 0.$$

The proof is a modified version of the one for finite dimensions.

LEMMA. *Suppose $\xi: \Gamma \to \mathscr{F}(E, F)$ is a continuous map of a topological space Γ into Fredholm operators from E to F. Then there exists a point $b \in \Gamma$ and a neighborhood U of b such that for $x \in U$*, dim Ker $\xi(x)$ *is constant.*

PROOF. Let X_s be the closed subset of X defined by

$$X_s = \{x \in \Gamma \mid \dim \operatorname{Ker} \xi(x) \geqq s\}.$$

Let s_0 be the smallest s so that $\sim X_s \neq \varnothing$ for $s > s_0$ or equivalently $X_s \neq X_{s_0} = X$, for $s > s_0$. Let $b \in X$ be chosen so that dim Ker $\xi(b) = s_0$, and U be an open neighborhood of b with the property that dim Ker $\xi(x) \leqq$ dim Ker $\xi(b)$ for $x \in U$.

THEOREM 2. *Suppose $f: M \to N$ is a C^1 injective Fredholm map between two Banach manifolds. Then the set $S(f) = \{x \mid Df_x$ not injective$\}$ of singularities of f is nowhere dense in M.*

PROOF. Clearly $S(f)$ is closed. Suppose $S(f)$ contains an open set, then we may work locally and assume M is open in E, N open in F and $S(f)$ contains an open ball B about one of its points, so $B \subset S(f)$. Define $\xi: B \to \phi(E, F)$ by $\xi(x) = Df_x$. From the lemma it follows that there is an open set $A \subset B$ on which dim Ker $\xi(x) = s > 0$.

Applying Theorem 1 we obtain a representative of f which contradicts the fact that f is an injection.

THEOREM 3. *Let $f: M \to N$ be a proper $\mathscr{F}_0$ map between two Banach manifolds with N connected. If f is injective then it is surjective.*

PROOF. By Theorem 2 there is a point x_0 with Df_{x_0} a linear injection. Thus $y=f(x_0)$ is a regular value, $f^{-1}(y)=x_0$, and $\gamma(f)=1$.

The above proposition is clearly false if we take maps which are not proper. For example the map which takes the plane homeomorphically onto the two-sphere minus the north pole is Fredholm of index zero, but not onto the sphere, however the map is open. This is summarized in the following.

THEOREM 4 (INVARIANCE OF DOMAIN). *Let $f: M \to N$ be a locally injective $\mathscr{F}_0$ map. Then f is an open map.*

PROOF. Let $y=f(x)$. We must show there is a neighborhood of x which is mapped to a neighborhood of y. Choose a connected open neighborhood U of x in M such that $f|\bar{U}$ is proper and injective. Then $y \notin f(\partial U)$. Let V be the connected component of $N-f(\partial U)$ containing y. Since $f|\bar{U}$ is proper, $f(\partial U)$ is closed and so V is open in N. Moreover $fU \subset V$ by connectedness. Finally, since $\bar{U} \cap f^{-1}V=U$, the map $f|U: U \to V$ is proper. By Theorem 3 it is therefore onto. Hence fU $(=V)$ is open.

THEOREM 5 (NONLINEAR FREDHOLM ALTERNATIVE). *Let $f: M \to N$ be a locally injective proper $\mathscr{F}_0$ map. If N is connected then f maps M onto N and is a covering map.*

PROOF. By Theorem 4, $f(M)$ is open in N. Since f is proper it is also closed and thus all of N. From the fact that f is an open map it follows that it is a local homeomorphism and then the properness of f implies that N is locally evenly covered by f.

THEOREM 6. *If $f: M \to N$ is as in Theorem 5 with M connected and N simply connected then f is a homeomorphism.*

PROOF. Follows from a standard covering space argument.

The above theorems can be proven by passing to local coordinates and using results already known for compact fields, e.g. Granas **[4]**, however the results illustrate the natural beauty and simplicity of degree theory arguments. We conclude this section with a generalization of a lemma of M. Hirsch to Banach manifolds. Using it one can, in special cases, prove the classical Schauder fixed point theorem. The proof of the following is essentially the same as in finite dimensions.

THEOREM 7. *Let X be a Banach manifold with boundary ∂X (also a smooth Banach manifold). Then there exists no proper $\mathscr{F}_0$ map $f: X \to X$ with $f(X) \subset \partial X$ and $f|\partial X=\mathrm{id}_{\partial X}$; i.e., ∂X is not a proper $\mathscr{F}_0$ retract of X.*

Bibliography

1. J. Callahan, *Intersection theory for infinite dimensional manifolds*, Thesis, New York University, 1967.

2. D. Elworthy and A. Tromba, *Differential structures and Fredholm maps on Banach manifolds*, Proc. Sympos. Pure Math., vol. 15, Amer. Math. Soc., Providence, R.I., 1970, pp. 45–93.

3. ———, *Degree theory on Banach manifolds*, Proc. Sympos. Pure Math., vol. 18, Amer. Math. Soc., Providence, R.I., 1970.

4. A. Granas, *The theory of compact vector fields and some of its applications to the topology of functional spaces*. I, Rozprawy Mat. **30** (1962), 93 pp. MR **26** #6743.

5. K. K. Mukherjea, *Coincidence theorems for infinite dimensional manifolds*, Bull. Amer. Math. Soc. **74** (1968), 493–496. MR **36** #5965.

6. U. Koschocke, *Infinite dimensional K-theory and characteristic classes of Fredholm bundle maps*, Proc. Sympos. Pure Math., vol. 15, Amer. Math. Soc., Providence, R.I. 1970.

7. R. Palais, *Lusternik-Schirlman theory on Banach manifolds*, Topology **5** (1966), 115–132. MR **41** #4584.

8. ———, *Homotopy theory of infinite dimensional manifolds*, Notes, Brandeis University, Waltham, Mass., 1967.

9. S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–866.

10. H. Widom, *Integral equations*, Van Nostrand, Princeton, N.J., 1968.

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