ON CONVOLUTIONS WITH THE MÖBIUS FUNCTION

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Abstract. By using the results of [6], it is proved that for an extensive class of increasing functions \( h \),

\[
(*) \quad \sum_{1 \leq d \leq x} \frac{\mu(d)}{d} h\left( \frac{x}{d} \right) \sim xh'(x) \quad \text{as } x \to \infty
\]

where \( \mu \) denotes the Möbius function. This result incidentally settles affirmatively Remark (iii) of [6], and refines the Tauberian Theorem 2 of that paper. It is also shown that one type of condition imposed in [6] is necessary to the truth of the cited Theorem 2, at least if some sort of quasi-Riemann hypothesis is true. Nevertheless, examples are given to show that on the one hand \((*)\) may be true for functions not covered by the first theorem of this paper, and on the other that some sort of nonnaïve condition on a function \( h \) is necessary to ensure the truth of \((*)\).

Much of this note, as will be evident, is in the nature of an interesting addendum to [6]; had I had the wit to notice it earlier, it should of course have been incorporated there.

Throughout, \( x \) denotes a real variable and \( s = \sigma + it \) a complex variable, \( \sigma, t \) real. All sums begin at 1. Given a suitable function \( g \), \( \mathcal{I} \) will denote the operator defined by \( \mathcal{I}g(y) = \int_1^y (g(x)/x) \, dx \), and \( \mathcal{I}^r \) the \( r \)th iterate of \( \mathcal{I} \). \( \mu \) is the Möbius function and \( N(x) = \sum_{n \leq x} (\mu(n)/n) \). \( \zeta(s) \) is the Riemann zeta-function, \( \gamma \) is Euler's constant. When \( h(x) \) is constant, the right side of \((*)\) is to be interpreted as equal to \( o(1) \).

For convenience we state here the main results of [6] which will be used in the sequel:

**Theorem A.** Let \( f(x) \) be any function bounded and integrable in every finite subinterval of \([1, \infty)\) which satisfies

\[
(1) \quad \sum_{\nu \mid x} f \left( \frac{x}{\nu} \right) = xg(x) + o(x^2g'(x)),
\]

where \( g(x) \) is a positive, twice continuously differentiable function, defined...
on \([1, \infty)\) such that

(i) \(g'(x) > 0\) for \(x \in (1, \infty)\),

(ii) \(xg'(x)\) is nonincreasing from some point on,

(iii) for some positive integer \(k\), \(x(\log x)^k g'(x) = u(x)\) is nondecreasing from some point on, and \(\lim \inf_{x \to \infty} u(x) = \infty\).

Then

\[
\int_1^x \frac{f(t)}{t^2} \, dt \to g(x) - \sum_{v \leq x \atop v \in \mathbb{Z}} \frac{\mu(v)}{v} g\left(\frac{x}{v}\right) + o(xg'(x))
\]

as \(x \to \infty\).

(This is Theorem 2 of [6].)

THEOREM B. If \(f\) and \(g\) are as in Theorem A, then

\[
\int_1^x \frac{f(t)}{t^2 g'(t)} \, dt = x + o(x).
\]

(This is equation (19) of [6]; if \(f\) is nondecreasing, one then easily deduces \(f(x) \sim x^2 g'(x)\) as in Theorem 1 of [6].)

We now state

THEOREM 1. Let \(h(x)\) be a positive function which has the property that there exists a nonnegative integer \(r\) such that \(h \in C^{r+2}(1, \infty)\) and \(h(x) = \mathcal{F}^r g(x)\), where \(g\) satisfies (i), (ii), (iii) of Theorem A; then

\[
\sum_{d < x} \frac{\mu(d)}{d} g\left(\frac{x}{d}\right) \sim xh'(x) \quad \text{as} \quad x \to \infty.
\]

PROOF. By induction. For convenience, we define \(g(x) = 0\) for \(x < 1\). Suppose first the theorem were true for some \(k\), \(k \geq 1\). Let \(G(x) = \mathcal{F}^k g(x)\). Then (since \(k \geq 1\)) \(G\) is clearly unbounded, and by hypothesis

\[
\sum_{d < x} \frac{\mu(d)}{d} G\left(\frac{x}{d}\right) \sim xG'(x) \quad \text{as} \quad x \to \infty,
\]

and so

\[
\sum_{d < x} \frac{\mu(d)}{d} \int_1^x \frac{G(t)}{t} \, dt = \int_1^x \frac{1}{t} \sum_{d < t} \frac{\mu(d)}{d} G\left(\frac{t}{d}\right) \, dt
\]

\[
= \int_1^x G'(t)(1 + o(1)) \, dt = G(x) + o(G(x)),
\]

which establishes the theorem for \(r = k + 1\). It remains to prove the theorem for \(r = 0\) and \(r = 1\). For \(r = 0\), \(h(x) = \mathcal{F}^0 g(x) = g(x)\). Let

\[
F(x) = \sum_{d < x} \frac{\mu(d)}{d} g\left(\frac{x}{d}\right).
\]
Then
\[ \sum_{d \leq x} \frac{x}{d} F\left(\frac{x}{d}\right) = xg(x) \]
and so, by Theorem A,
\[ \int_1^x \frac{F(t)}{tg(t)} \, dt = g(x) - \gamma \sum_{\nu \leq x} \frac{\mu(\nu)}{\nu} g\left(\frac{x}{\nu}\right) + o(xg'(x)); \]
and, by Theorem B,
\[ \int_1^x \frac{F(t)}{tg(t)} \, dt = x + o(x). \]
(If \( xF(x) \) were monotone, the conclusion, as remarked previously, would now follow almost immediately; unfortunately this need not be the case.)

Furthermore the hypotheses on \( g \) imply (see Lemma 1 of [6] for the easy deduction)
\[ \lim_{x \to \infty} \frac{g''(x)}{g'(x)} = -1 \quad \text{and also that} \]
\[ \frac{g''(x)}{g'(x)} = \frac{(d/dx)(xg'(x))}{g'(x)} - 1 \leq -1, \]
from some point on. Hence \( xg''(x)/g'(x) \) is eventually nondecreasing and so eventually has a nonnegative derivative, a.e.

Now, integrating by parts in (5), after using (4), gives
\[ \int_1^x \frac{F(t)}{tg(t)} \, dt = \frac{1}{g'(x)} \int_1^x \frac{F(t)}{t} \, dt + \int_1^x \frac{g''(t)}{(g'(t))^2} \int_1^t \frac{F(u)}{u} \, du \, dt \]
\[ = g(x) - \frac{\gamma F(x)}{g'(x)} + o(x) + \int_1^x \frac{g''(t)}{(g'(t))^2} (g(t) - \gamma F(t) + o(tg'(t))) \, dt \]
\[ = x - \frac{\gamma F(x)}{g'(x)} - \gamma \int_1^x \frac{F(t)g''(t)}{(g'(t))^2} \, dt + o(x) \]
(cf. treatment of equation (15) in [6]).

And so (5) yields
\[ \frac{F(x)}{g'(x)} = -\int_1^x \frac{F(t)g''(t)}{(g'(t))^2} \, dt + o(x). \]

But
\[ \int_1^x \frac{F(t)g''(t)}{(g'(t))^2} \, dt = \int_1^x \frac{F(t)}{tg(t)} \frac{t}{g'(t)} \, dt \]
\[ = \frac{xg''(x)}{g'(x)} \int_1^x \frac{F(t)}{tg(t)} \, dt - \int_1^x \frac{d}{dt} \left(\frac{tg''(t)}{g'(t)}\right) \int_1^t \frac{F(u)}{ug'(u)} \, du \, dt + O(1), \]
and on substituting this in (8), and using (6) and (5), we get
\[
F(x) = x + o(x) + \int_1^x \frac{d}{dt} \left( \frac{tg'(t)}{g'(t)} \right) (t + o(t)) \, dt
\]
(9)
\[
= x + o(x) + O\left( \int_1^x \frac{d}{dt} \left( \frac{tg'(t)}{g'(t)} \right) \right).
\]
However, by (6),
\[
\int_1^x t \frac{dg'(t)}{g'(t)} = \frac{x^2 g''(x)}{g'(x)} - \int_1^x \frac{tg'(t)}{g'(t)} \, dt + O(1)
\]
\[
= -x + o(x) + \int_1^x (1 + o(1)) \, dt + O(1) = o(x),
\]
and substituting this in (9) gives
\[
(10) \quad F(x) \sim x g'(x)
\]
as claimed. This proves the case \( r=0 \). The case \( r=1 \) now follows on substituting (10) in (4) to obtain
\[
(11) \quad \int_1^x \frac{F(t)}{t} \, dt = g(x) - \gamma x g'(x) + o(x g'(x)),
\]
and noting that
\[
\sum_{d \leq x} \frac{\mu(d)}{d} \int_1^x \frac{g(t)}{t} \, dt = \int_1^x \frac{F(t)}{t} \, dt,
\]
and that the conditions on \( g \) imply \( x g'(x) = o(g(x)) \) as \( x \to \infty \).

Remarks. (a) To prove the results of [6] used above, it was necessary to invoke a fairly strong form of the prime number theorem: \( N(x) = o((\log x)^{-k}) \) for every \( k > 0 \). In the other direction, for \( k \) a positive integer, \( (\log x)^k \) satisfies the conditions placed on \( h(x) \) in Theorem 1 (with \( g(x) = \log x \)), and so we get, from Theorem 1,
\[
\sum_{d \leq x} \frac{\mu(d)}{d} \left( \frac{x^k}{d} \right) \sim k(\log x)^{k-1}
\]
for every positive integer \( k \) (which is, of course, also deducible directly from \( N(x) = o((\log x)^{-k}) \) for every \( k > 0 \)).

(b) The conclusion of Theorem A can now be amended to read:
\[
(12) \quad \int_1^x \frac{f(t)}{t^2} \, dt = g(x) - \gamma x g'(x) + o(x g'(x)).
\]
g(\( x \)) was assumed for this theorem to be twice continuously differentiable.
In practice, however, $g$ is usually analytic. Suppose we assume $g$ is analytic; under what subsidiary conditions on $g$ can (12) be replaced by a similar asymptotic expansion with a desired number of terms for

$$\int_1^x \frac{(F(t)/t^a)}{dt}?$$

Condition (iii) of Theorem A seems somewhat unnatural; while it may be somewhat ameliorated by using stronger results from prime number theory than used in [6] (see Remark (iv) of that paper), the real question is whether any condition in addition to (i) and (ii) is necessary for the truth of Theorem A. Assuming a "quasi-Riemann hypothesis", the answer is "yes", as Theorem 2 below shows.

**Theorem 2.** Suppose some sort of quasi-Riemann hypothesis holds; i.e. suppose $\zeta(s) \neq 0$ for some strip $b < \sigma \leq 1$ ($b \geq \frac{1}{2}$). Then there is a function $k(x)$ satisfying (i) and (ii) of Theorem A which is analytic in $(1, \infty)$ and for which the conclusion of Theorem A (in the form of (12)) is false. A similar contradiction is even easier to obtain in Theorem A’s original form.

**Proof.** Let $a \in (b, 1)$. Define $a(x) = 1 - e^{-x}$. Then $k_a(x)$ satisfies (i), (ii) as is easily checked. By the hypothesis on $\zeta$ we have that, as $x \to \infty$,

$$\int_1^x \frac{N(t)}{t} dt = -1 + o(x^{a-1}),$$

and

$$\int_1^x \frac{N(t)}{t^a} dt \text{ converges to } \frac{1}{(a-1)\zeta(a)}.$$

(These results arise from using partial summation on

$$\sum_{d \leq x} \frac{\mu(d)\log d}{d} \quad \text{and} \quad \sum_{d \leq x} \frac{\mu(d)}{d^a} = \sum_{d \leq x} \frac{\mu(d)}{d} d^{1-a},$$

and taking note of results usually stated in the literature for $b = \frac{1}{2}$ (the Riemann hypothesis), but whose analogues for other values of $b$ are immediate, see e.g. [5], [7, p. 315].)

Let $f(x) = (1-a)x^a \int_1^x \frac{N(t)}{t^a} dt$. Then

$$\sum_{d \leq x} \frac{\mu(d)\log d}{d} \quad \text{and} \quad \sum_{d \leq x} \frac{\mu(d)}{d^a} = \sum_{d \leq x} \frac{\mu(d)}{d} d^{1-a},$$

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since $\sum_{d\leq u} (1/d)N(u/d) = 1$, and $N(x) = 0$ for $x < 1$. If the conclusion expressed by (12) held for all $k_\alpha(x)$, then we would have

$$
\int_1^x ft^\alpha \text{d}t = (1 - \alpha) \int_1^x t^{\alpha - 2} \int_1^t N(u) \text{d}u \text{d}t
$$

$$(15) \quad = 1 - x^{\alpha - 1} - \gamma(1 - \alpha)x^{\alpha - 1} + o(x^{\alpha - 1}) \quad \text{for all } a, b < a < 1. $$

But integrating by parts in (15) gives

$$
- x^{\alpha - 1} \int_1^x \frac{N(u)}{u^\alpha} \text{d}u - \int_1^x \frac{N(t)}{t} \text{d}t = 1 - (1 + \gamma(1 - a))x^{\alpha - 1} + o(x^{\alpha - 1})
$$

and so by (13), as $x \to \infty$,

$$
\int_1^x \frac{N(u)}{u^\alpha} \text{d}u = 1 + \gamma(1 - a) + o(1) \quad \text{for all } a, \frac{1}{2} \leq b < a < 1;
$$

or by (14),

$$
(16) \quad \frac{1}{(a - 1)\zeta(a)} = 1 + \gamma(1 - a) \quad \text{for all } a, \frac{1}{2} \leq b < a < 1.
$$

For a given fixed value of $a$, (16) might be shown false by ad hoc computation, however, this would not suffice to prove the theorem for any quasi-Riemann hypothesis; instead, we simply argue as follows: (16) would imply

$$
(17) \quad \frac{\zeta(a) - (1/(a - 1)) - \gamma}{a - 1} = \frac{\gamma^2}{1 - \gamma(a - 1)} , \quad \text{for all } a, \frac{1}{2} \leq b < a < 1.
$$

Letting $a \to 1$ in (17) would give

$$
\lim_{a \to 1} \frac{\zeta(a) - (1/(a - 1)) - \gamma}{a - 1} = \gamma^2 .
$$

But the limit on the left is known to equal

$$
-\gamma_1 \equiv - \lim_{n \to \infty} \left( \sum_{v=1}^{n} \frac{\log v}{v} - \frac{1}{2}(\log n)^2 \right) = 0.07281 + \neq \gamma^2
$$

which is the desired contradiction. (Two proofs of the form taken by the coefficients of the Taylor expansion of $\zeta(s-1)/(s-1)$ about 1 may be found in [1]; the result was known to Hardy in 1912, and is no doubt much older still. The computation of $-\gamma_1$ was made by Wilton [8] from a different expression for that constant.)
We close with two examples further illuminating the relationship (*).

Example 1. There is a nondecreasing function \( h \) for which (ii) does not hold but (*) still does.

Proof. Take \( h(x) = \int_1^x \frac{N(t)}{t} \, dt \). It is easy to see that \( |N(x)| \leq 1 \) for all \( x \geq 1 \) [4, p. 583] whence \( h \) is nondecreasing; clearly \( xh'(x) = 1 + N(x) \) is not monotone decreasing. However,

\[
\sum_{d \leq x} \frac{\mu(d)}{d} h\left( \frac{x}{d} \right) = \sum_{d \leq x} \frac{\mu(d)}{d} \log \left( \frac{x}{d} \right) + \sum_{d \leq x} \frac{\mu(d)}{d} \int_1^{x/d} \frac{N(t)}{t} \, dt
\]

\[
= 1 + o(1) - \sum_{d \leq x} \frac{\mu(d)}{d} \left( \sum_{m \leq x/d} \frac{\mu(m) \log m}{m} - \sum_{m \leq x/d} \frac{\mu(m)}{m} \log \left( \frac{x}{d} \right) \right)
\]

\[
= 1 + o(1) - \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{m \leq x/d} \frac{\mu(m) \log m}{m} + \log x \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{m \leq x/d} \frac{\mu(m)}{m}
\]

\[
= 1 + o(1),
\]

since \( \sum_{d \leq x} (\mu(d) \log d/d) = -1 + o(1) \) (e.g. [4, §158]),

\[
\sum_{d \leq x} \frac{\mu(d)}{d} \sum_{m \leq x/d} \frac{\mu(m) \log m}{m} = o(1), \quad \text{and} \quad \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{m \leq x/d} \frac{\mu(m)}{m} = o\left( \frac{1}{\log x} \right)
\]
as \( x \to \infty \); these last two being easily proved by a technique going back to Dirichlet and embodied in the argument in [4, p. 685].

Example 2. There is a function \( h \) such that \( h(x) \sim C \) as \( x \to \infty \), \( C \) a constant, but \( \sum_{d \leq x} \frac{\mu(d)}{d} h(x/d) = o(1) \) as \( x \to \infty \).

Proof. It is known that there are \((L)\)-summable series which are not convergent; i.e. there is a sequence \( \{a_n\} \) such that \( \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} d_n a_d \) exists but \( \sum a_n \) does not converge. (This is stated in [3]; a proof may be obtained by applying the Silverman-Toeplitz Theorem to the inverse of the \((L)\)-transformation.) It is also known that \((L)\)-summability implies \((C, 1)\)-summability [3]. Let

\[
L(x) = \frac{1}{x} \sum_{n \leq x} \sum_{d \mid n} da_d = \frac{1}{x} \sum_{d \leq x} \sum_{m \leq x/d} ma_m
\]

where \( \{a_n\} \) is such a sequence. Then

\[
\sum_{d \leq x} \frac{\mu(d)}{d} L\left( \frac{x}{d} \right) = \frac{1}{x} \sum_{m \leq x} ma_m,
\]
lim_{x \to \infty} I(x) exists, \sum a_n is (C, 1)-summable but not convergent, and so \sum_{m \leq x} ma_m \neq o(x) as x \to \infty (e.g. [2, Theorem 65]).

REMARK. No explicit example of such a sequence \{a_n\} seems to be known.

REFERENCES

5. J. E. Littlewood, Quelques conséquences de l'hypothèse que la fonction \(\zeta(s)\) de Riemann n'a pas de zéros dans le demi-plan \(R(s) > \frac{1}{2}\), Comptes Rendus 154 (1912), 263–266.
8. J. R. Wilton, A note on the coefficients of the expansion of \(\zeta(s, x)\) in powers of \(s - 1\), Quart. J. Math. 50 (1927), 329–332.

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