THE LOCATION OF THE ZEROS OF THE DERIVATIVES OF FABER POLYNOMIALS

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Abstract. We prove a property of Faber polynomials which supports a conjecture concerning the zeros of Faber polynomials.

1. Introduction. Let $K$ be a compact set consisting of more than one point having a connected and simply connected complement, $\tilde{K}$. Let $f(z)$ be the schlicht function mapping $\tilde{K}$ onto the exterior of the circle $\Sigma_w = \{w:|w|=1\}$, normalized so that

$$f(z) = z + a_0 + a_1z^{-1} + \cdots.$$ 

This normalization can always be achieved by changing units appropriately in the $z$ plane, and we assume that this is done.

The polynomial part of $(f(z))^n$, $F_n(z)$, is called the Faber polynomial of degree $n$ associated with $K$. There is a conjecture that for each $n$, $n=1, 2, \cdots$, the zeros of $F_n(z)$ lie in $H(K)$, the convex hull of $K$. I will show in this paper that $H(K)$ contains all the zeros of $F'_n(z)$, $n=1, 2, \cdots$.

2. Let

$$z = g(w) = w + b_0 + b_1w^{-1} + \cdots$$

be the inverse function of $w=f(z)$. This is a schlicht function mapping the exterior of $\Sigma_w$ onto $\tilde{K}$. Since $g(w)$ is bounded in $\{w:1<|w|<R\}$, $R>1$, $\lim_{r \to 1^+} g(re^{i\theta})$ exists for almost all $0 \leq \theta \leq 2\pi$. Let $g(\theta)$ be a function defined as this limit, where it exists, and as a value lying in $K$ for other values of $\theta$. Then, because of the schlichtness of $g(w)$, all the values of $g(\theta)$ lie in $K$. Finally, using Lebesgue’s theorem of bounded convergence, we find that

$$\frac{1}{2\pi} \int_0^{2\pi} g(\theta)e^{-in\theta} d\theta = b_n, \quad n = 0, -1, \cdots,$$

(2)

$$= 1, \quad n = 1,$$

$$= 0, \quad n = 2, 3, \cdots.$$
3. Faber [1] showed that

\[ \frac{g'(w)}{g(w) - z} = \sum_{n=0}^{\infty} F_n(z) w^{n+1} \]

and from this we easily obtain

\[ \frac{1}{g(w) - z} = \sum_{n=1}^{\infty} n w^n. \]

4. Our object now is to get another representation for \( F_n'(z) \) by regarding \( (g(w) - z)^{-1} \) as the reciprocal of a power series, namely

\[ \frac{1}{g(w) - z} = \frac{1}{w + (b_0 - z) + b_{-1}w^{-1} + \cdots}. \]

For a power series \( \sum_{n=0}^{\infty} c_n z^n \), it is readily found that if \( \sum_{n=0}^{\infty} d_n z^n = (\sum_{n=0}^{\infty} c_n z^n)^{-1} \), then

\[ d_n = (-1)^n \begin{vmatrix} c_1 & c_0 & 0 & \cdots \\ c_2 & c_1 & c_0 & \cdots \\ & & \ddots & \ddots \\ & & & c_0 \end{vmatrix} \begin{vmatrix} c_n \\ \vdots \\ c_0 \end{vmatrix}. \]

We can apply (6) to (5) and use (4) to obtain

\[ \frac{F_n'(z)}{n + 1} = (-1)^n \begin{vmatrix} b_0 - z & 1 & 0 & \cdots \\ \vdots & b_{-1} & b_0 - z & 1 & \cdots \\ & & \ddots & \ddots & \ddots \\ & & & \cdots & 1 \end{vmatrix} \begin{vmatrix} b_{-(n-1)} \\ \vdots \\ b_0 - z \end{vmatrix}. \]

Thus we see that the zeros of the derivatives of Faber polynomials are of the eigenvalues a type of matrix known as a Toeplitz matrix.

5. If \((e_{p,q})\) is an \(n\)th order matrix and if \(\lambda\) is an eigenvalue, then there is a nondegenerate vector \((x_1, \ldots, x_n)\) such that

\[ \sum_{q=1}^{n} e_{p,q} x_q = \lambda x_p, \quad p = 1, \ldots, n. \]
From this we obtain

$$\sum_{p=1}^{n} \sum_{q=1}^{n} e_{p,q} x_{p} x_{q} = \lambda \sum_{p=1}^{n} |x_{p}|^2.$$  

Now, in the case of (7), we have $e_{p,q} = b_{q-p}$, if we understand that $b_1 = 1$ and $b_n = 0$ for $n = 2, 3, \ldots$. Thus if $z_0$ is a zero of $F_{n+1}'(z)$ we find that there is a nondegenerate vector $(x_1, \ldots, x_n)$ such that

$$\sum_{p=1}^{n} \sum_{q=1}^{n} b_{q-p} x_{q} x_{p} = z_0 \sum_{p=1}^{n} |x_{p}|^2.$$  

Using (2) this becomes

$$\frac{1}{2\pi} \int_{0}^{2\pi} g(\theta) |x_1 + x_2 e^{-i\theta} + \cdots + x_n e^{-i(n-1)\theta}|^2 \, d\theta$$  

$$= z_0 \frac{1}{2\pi} \int_{0}^{2\pi} |x_1 + x_2 e^{-i\theta} + \cdots + x_n e^{-i(n-1)\theta}|^2 \, d\theta.$$  

This shows that $z_0$ is a weighted arithmetic average of the values in the essential range of $g(\theta)$, and thus lies in the convex hull of the essential range of $g(\theta)$. We have already noted that the values of $g(\theta)$ lie in $K$, and thus the convex hull of the essential range of $g(\theta)$ lies in the convex hull of $K$, and our proof that $z_0$, a zero of $F_{n+1}'(z)$, lies in the convex hull of $K$ is complete. The results of this paper now permit us to apply the results of [2] to the zeros of the derivatives of Faber polynomials. In addition, the methods of [2] can be applied to obtain results on the asymptotic distribution of the Faber polynomials themselves.

REFERENCES