ON MODULAR COMPLEMENTED AND ANNIHILATOR ALGEBRAS

T. HUSAIN¹ AND PAK-KEN WONG²

Abstract. Let $A$ be a modular complemented algebra. We give some necessary and sufficient conditions for some special classes of algebras $A$ to be annihilator or dual algebras.

1. Introduction. Modular complemented algebras were introduced by Yood in [7], where he developed a structure and ideal theory for these algebras. In this paper our aim is to investigate the relationship among modular complemented, annihilator and dual algebras.

Let $A$ be a modular complemented algebra. We show that if $I + I^\perp = A$ for all minimal (left, right) ideals $I$ of $A$, then $A$ is an annihilator algebra if and only if every nonzero left ideal of $A$ contains an element with right adjoint. If $J + J^\perp = A$ for all (left, right) ideals $J$ of $A$, and $A$ is a left or right annihilator algebra, then it is a dual algebra.

2. Notation and preliminaries. Let $A$ be an algebra over the complex field $C$ which is a pre-Hilbert space in the inner product $(x, y)$. $A$ is called a right-modular complemented algebra if it satisfies the following conditions:

(a) the product $xy$ is continuous in each variable separately,
(b) every right or left ideal $I$ for which $I^\perp = (0)$ is dense in $A$,
(c) the intersection of all closed modular maximal right ideals is $(0)$ and $M^\perp$ is a right ideal for each closed modular maximal right ideal $M$.

Remark. From (c) it follows that every right-modular complemented algebra is semisimple.

Similarly we define a left-modular complemented algebra. It has been shown [7] that an algebra $A$ is right-modular complemented if and only

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if it is left-modular complemented. In this case, \( K^\perp \) is a right (left) ideal for any right (left) ideal \( K \) of \( A \) (see [7, p. 262, Theorem 2.1]).

Let \( A \) be an algebra which is a pre-Hilbert space. As in [7], we say that \( u^l \) is a left adjoint for \( u \) if \((ux, y) = (x, u^l y)\) for all \( x, y \in A \). If \( A \) is semi-simple, then the left adjoint is unique if it exists. Let \( u, v \in A \). If \( u^l, v^l \)
eq 0 exists then \((uv)^l\) exists and \((uv)^l = v^l u^l\). If \( u = u^l \), then \( u \) is said to be left selfadjoint. Similarly \( u^r \) is called the right adjoint if \((xu, y) = (x, yu^r)\) for all \( x, y \in A \).

For any subset \( E \) in an algebra \( A \), let \( L(E) \) (\( R(E) \)) denote the left (right) annihilator of \( E \) in \( A \). Suppose \( A \) is a topological algebra then we call \( A \) an annihilator algebra provided that, for every closed left ideal \( M \) and for every closed right ideal \( N \), we have \( R(M) = (0) \) if and only if \( M = A \) and \( L(N) = (0) \) if and only if \( N = A \). If \( M = L(R(M)) \) and \( N = R(L(N)) \), then \( A \) is called a dual algebra.

In this paper, all algebras and linear spaces are over the complex field \( C \). The results of this paper heavily depend on [6] and [7].

3. Characterizations of annihilator and dual algebras. It is known that a modular complemented algebra need not be an annihilator algebra (see [7, p. 263]). In our first result, we give a characterization for annihilator modular complemented algebras.

Let \( A \) be a topological algebra. Then \( A \) is called a left annihilator algebra if for every closed right ideal \( M \) of \( A \), \( L(M)^\perp = (0) \) and \( L(A) = (0) \). Similarly we define a right annihilator algebra.

Theorem 3.1. Let \( A \) be a modular complemented algebra such that \( I^\perp \cap I^\perp = A \) for all minimal (left, right) ideals \( I \) of \( A \). Then the following statements are equivalent:

(i) \( A \) is an annihilator algebra.
(ii) \( A \) is a right annihilator algebra.
(iii) Every element of the socle \( S \) of \( A \) has a right adjoint.
(iv) Every nonzero left ideal of \( A \) contains an element with right adjoint.

Proof. (i)\( \Rightarrow \) (ii). This is clear.

(ii)\( \Rightarrow \) (iii). Suppose \( A \) is a right annihilator algebra. Let \( I \) be a minimal left ideal of \( A \). Since \( A = I \oplus I^\perp \), it follows easily that \( I^\perp \) is a maximal closed left ideal. Thus by [1, p. 568, Theorem 3.4], \( I^\perp \) is a modular maximal left ideal. Hence by the proof of [6, p. 44, Theorem 4.5] each element of \( I \) has a right adjoint and so does \( S \).

(iii)\( \Rightarrow \) (iv). Let \( M \) be a nonzero left ideal of \( A \). Since \( S \) is dense in \( A \) (see the proof of [7, p. 262, Theorem 2.1]), we have \( S \cap M \neq (0) \). Therefore (iv) follows from (iii).
(iv)⇒(ii). Assume (iv). Let $T$ be the set given in the proof of [7, p. 262, Theorem 2.1]. Then $T$ is a dense right ideal of $A$ and each element in $T$ has a left adjoint. Let $J$ be a proper closed left ideal of $A$. Then $J\parallel_A(0)$. Let $x$ be an element in $J\parallel_A$ which has a right adjoint $x^\ast$. Let $z \in J$ and $y \in T$. Then $(yx, z) = (x, y^\ast z) = 0$. Since $T$ is dense in $A$, we have $(Ax, z) = (0)$ and so $(A, zx^\ast) = (0)$ which gives $zx^\ast = 0$. Hence $Jx^\ast = 0$ and so $R(J)\parallel_A(0)$. Therefore $A$ is a right annihilator algebra.

(ii)⇒(i). Suppose (ii) holds. Let $I$ be a minimal left ideal of $A$. By the argument in (ii)⇒(iii), $I\parallel_A$ is a modular maximal left ideal. Hence by the proof of [7, p. 263, Theorem 2.3], each element of $I$ has a left adjoint and so does $S$. Hence by a similar argument in the proof (iv)⇒(ii), we can show that $A$ is a left annihilator algebra. This completes the proof of the theorem.

**Remark.** A similar form of Theorem 3.1 holds for left annihilator algebras.

**Theorem 3.2.** Let $A$ be a modular complemented algebra such that $J\otimes I\parallel_A A$ for all closed (left, right) ideals $J$ of $A$. Then the following statements are equivalent:

(i) $A$ is a dual algebra.

(ii) $A$ is a left or right annihilator algebra.

**Proof.** (i)⇒(ii). This is clear. (ii)⇒(i). Assume (ii). Then by Theorem 3.1 and its remark, $A$ is an annihilator algebra. Let $I$ be a proper closed right ideal of $A$. Since $I\parallel_A(0)$ and since the socle of $A$ is dense in $A$, it follows that $I\parallel_A$ contains a minimal right ideal $J$. By the proof of Theorem 3.1, $J\parallel_A$ is a closed modular maximal right ideal and clearly $I\parallel_J I$. Let $\{M_\lambda: \lambda \in \Lambda\}$ be the family of all closed modular maximal right ideals of $A$ containing $I$. Then $\{M_\lambda: \lambda \in \Lambda\}$ is not empty. Let $M = \bigcap \{M_\lambda: \lambda \in \Lambda\}$. We claim that $M = I$. In fact, suppose this is not so. Then there exists an element $x \in M$ such that $x \notin I$. Since $I\oplus I\parallel_A = A$, we can write $x = x_1 + x_2$ with $x_1 \in I$ and $x_2 \in I\parallel_A$. Hence $x_2 = x - x_1 \in I\parallel_A \cap M$ and so $I\parallel_A \cap M$ contains a minimal right ideal $J$ whose orthogonal complement $J\parallel_A$ is a closed modular maximal right ideal by the above argument. Therefore by the proof of [6, p. 44, Theorem 4.5], $J$, and hence $I\parallel_A \cap M$, contains a left selfadjoint minimal idempotent $e$. Since $eA \subset I\parallel_A$, we have $(1-e)A = (eA)\parallel_A I$. Therefore $(1-e)A \supset M$. But $e \in M$, a contradiction. Hence $M = I$. Since each $M_\lambda = R(L(M_\lambda))$, we have $I = R(L(I))$. Similarly we can show that $N = L(R(N))$ for all closed left ideals $N$ of $A$. Therefore $A$ is a dual algebra. This completes the proof.

A modular complemented algebra is said to be complete if it is complete in its inner product norm.
COROLLARY 3.3. Let $A$ be a complete modular complemented algebra. Then the conditions (i), (ii), (iii) and (iv) in Theorem 3.1 are equivalent to the statement that $A$ is a dual algebra.

PROOF. This follows from Theorems 3.1 and 3.2.

REMARK. It is well known that a proper $H^*$-algebra $A$ is dual. This result also follows from Corollary 3.3, since every element of $A$ has left and right adjoints.

If a modular complemented algebra is a normed algebra under its inner product norm, then we say that it is a normed modular complemented algebra.

THEOREM 3.4. Let $A$ be a complete normed modular complemented algebra. Then

(i) $A$ is a dense left ideal of a dual $B^*$-algebra $B$.

(ii) If $A$ is also a right ideal of $B$, then $A$ is a dual algebra.

PROOF. (i) For each $x \in A$, let $U_x$ be the operation of left multiplication by $x$ and $|U_x|$ the operator bound of $U_x$. Let $B$ be the closure of $\{U_x : x \in A\}$ in the uniform topology. Since $A$ is semisimple, the mapping $x \rightarrow U_x$ is one-one. Therefore we can identify $A$ as a subalgebra of $B$. Let $T$ be the algebraic sum of the $M^+$ as $M$ ranges over the set of all closed modular maximal right ideals of $A$. By the proof of [6, p. 44, Theorem 4.5], $T$ is dense in $A$, the left adjoint $a'$ exists for all $a \in T$ and $a' \in T$. Clearly $U_{a'}=(U_a)^*$ and so $T^*=T$ in $B$. Since $T$ is dense in $B$, it follows that $B=B^*$. Therefore $B$ is a $B^*$-algebra. Since the socle of $B$ contains $T$, it follows from [4, p. 222, Theorem 2.1] that $B$ is a dual algebra. For all $x, y \in A$, we have $\|xy\| \leq \|x\| \|y\|$. This shows that $A$ is a left ideal of $B$ and completes the proof of (i).

(ii) Now it follows from (i) that $A$ is a dense two-sided ideal of $B$. For example, a proper $H^*$-algebra has such a property. Let $M$ be a non-zero right ideal of $A$. Then $M$ contains a minimal idempotent $e$ of $A$. It is clear that $eB=eA$ and so $eA$ is a minimal right ideal of $B$. Hence by [5, p. 261, Lemma (4.10.1)], $eA$ contains a hermitian minimal idempotent $f \in B$. It is easy to see that $f \in A$ and $f$ is a left selfadjoint element in $A$. Since $f \in M$, it follows from Corollary 3.3 that $A$ is a dual algebra.

4. Hilbert algebras. A complex algebra $A$ is called a Hilbert algebra (see [3]) if it is a pre-Hilbert space with an involution $x \rightarrow x^*$ having the following properties:

(a) $(xy, z)=(y, x^*z)$ for all $x, y, z \in A$,

(b) $(x, y)=(y^*, x^*)$ for all $x, y \in A$.
(c) the operation of left multiplication $y \mapsto xy$ is continuous on $A$ for each $x \in A$,
(d) the set of elements of the form $xy$ is dense in $A$.
It follows easily from (a) and (b) that $(yx, z) = (y, zx^*)$ for all $x, y, z \in A$. It is easy to see that the right multiplication is also continuous. Since $\|x\| = \|x^*\|$, the involution is continuous.

A net $\{e_\alpha\}$ in a normed algebra $A$ is called an approximate identity if it is bounded and $x = \lim \alpha e_\alpha = \lim \alpha x e_\alpha$ for all $x \in A$.

The following result was proved by Yood for $B^*$-algebras (see [7, p. 267, Theorem 3.1]).

**Theorem 4.1.** Let $A$ be an infinite-dimensional Banach $^*$-algebra with an approximate identity $\{e_\alpha\}$. Then there does not exist an inner product for $A$ under which $A$ becomes a pre-$H^*$-algebra with the same involution.

**Proof.** Suppose otherwise. Let $|x| = (x, x)^{1/2}$ be the inner product norm and let $H$ be the completion of $A$ in the norm $|x|$. By the proof of [7, p. 267, Theorem 3.1], $H$ is a proper $H^*$-algebra and, for all $x \in A$, $|x| \leq k\|x\|$, where $k$ is a constant. Now it is easy to see that $\{e_\alpha\}$ is also an approximate identity in $H$. Since $H$ is reflexive, by [2, p. 855, Lemma 3.8], $H$ has an identity. Therefore by [7, p. 266, Corollary 2.7], $H$ is finite dimensional and so $A$, a contradiction. This completes the proof.

**Theorem 4.2.** Let $A$ be a Hilbert algebra. Then $A$ is a dual algebra if and only if $I = I^{\perp\perp}$ for all closed right ideals $I$ of $A$.

**Proof.** By the proof of [7, p. 268, Lemma 3.2], $x \in xA \cap Ax$. Therefore $L(A) = R(A) = (0)$. Suppose $A$ is a dual algebra. Let $I$ be a closed right ideal of $A$. Then by the proof of [7, p. 268, Lemma 3.2], we have

$$I = R(L(I)) = ((I^{\perp\perp})^{**})^{\perp\perp} = I^{\perp\perp}.$$  

Conversely suppose $I = I^{\perp\perp}$ for all closed right ideals $I$ of $A$. It is easy to see that $I^{\perp\perp} = R(L(I)) = I$. Let $J$ be a closed right ideal of $A$. Hence $J^\perp = (J^\perp)^\perp$ and so $J = J^{\perp\perp}$. It is now easy to see that $J = L(R(J))$. Therefore $A$ is a dual algebra.

**Corollary 4.3.** Let $A$ be a Hilbert algebra. If $I \otimes I = A$ for all closed right ideals $I$ of $A$, then $A$ is a dual algebra.

**References**


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DEPARTMENT OF MATHEMATICS, McMaster University, Hamilton, Ontario, Canada