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ON MODULAR COMPLEMENTED AND ANNIHILATOR ALGEBRAS

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ABSTRACT. Let A be a modular complemented algebra. We give some necessary and sufficient conditions for some special classes of algebras A to be annihilator or dual algebras.

1. **Introduction.** Modular complemented algebras were introduced by Yood in [7], where he developed a structure and ideal theory for these algebras. In this paper our aim is to investigate the relationship among modular complemented, annihilator and dual algebras.

Let A be a modular complemented algebra. We show that if $I+I^{\perp}=A$ for all minimal (left, right) ideals I of A, then A is an annihilator algebra if and only if every nonzero left ideal of A contains an element with right adjoint. If $J+J^{\perp}=A$ for all (left, right) ideals J of A, and A is a left or right annihilator algebra, then it is a dual algebra.

2. **Notation and preliminaries.** Let A be an algebra over the complex field C which is a pre-Hilbert space in the inner product (x, y). A is called a right-modular complemented algebra if it satisfies the following conditions:

(a) the product xy is continuous in each variable separately,

(b) every right or left ideal I for which $I^{\perp}=(0)$ is dense in A,

(c) the intersection of all closed modular maximal right ideals is (0) and $M^{\perp}$ is a right ideal for each closed modular maximal right ideal M.

REMARK. From (c) it follows that every right-modular complemented algebra is semisimple.

Similarly we define a left-modular complemented algebra. It has been shown [7] that an algebra A is right-modular complemented if and only

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if it is left-modular complemented. In this case, $K^{\perp}$ is a right (left) ideal for any right (left) ideal K of A (see [7, p. 262, Theorem 2.1]).

Let A be an algebra which is a pre-Hilbert space. As in [7], we say that u^l is a left adjoint for u if $(ux, y)=(x, u^l y)$ for all $x, y \in A$. If A is semi-simple, then the left adjoint is unique if it exists. Let $u, v \in A$. If u^l, v^l exists then $(uv)^l$ exists and $(uv)^l=v^l u^l$. If $u=u^l$, then u is said to be left selfadjoint. Similarly u^r is called the right adjoint if $(xu, y)=(x, yu^r)$ for all $x, y \in A$.

For any subset E in an algebra A, let $L(E)$ $(R(E))$ denote the left (right) annihilator of E in A. Suppose A is a topological algebra then we call A an annihilator algebra provided that, for every closed left ideal M and for every closed right ideal N, we have $R(M)=(0)$ if and only if $M=A$ and $L(N)=(0)$ if and only if $N=A$. If $M=L(R(M))$ and $N=R(L(N))$, then A is called a dual algebra.

In this paper, all algebras and linear spaces are over the complex field C. The results of this paper heavily depend on [6] and [7].

3. **Characterizations of annihilator and dual algebras.** It is known that a modular complemented algebra need not be an annihilator algebra (see [7, p. 263]). In our first result, we give a characterization for annihilator modular complemented algebras.

Let A be a topological algebra. Then A is called a left annihilator algebra if for every closed right ideal M of A, $L(M)\neq(0)$ and $L(A)=(0)$. Similarly we define a right annihilator algebra.

THEOREM 3.1. *Let A be a modular complemented algebra such that $I\oplus I^{\perp}=A$ for all minimal (left, right) ideals I of A. Then the following statements are equivalent*:

(i) *A is an annihilator algebra.*

(ii) *A is a right annihilator algebra.*

(iii) *Every element of the socle S of A has a right adjoint.*

(iv) *Every nonzero left ideal of A contains an element with right adjoint.*

PROOF. (i)$\Rightarrow$(ii). This is clear.

(ii)$\Rightarrow$(iii). Suppose A is a right annihilator algebra. Let I be a minimal left ideal of A. Since $A=I\oplus I^{\perp}$, it follows easily that $I^{\perp}$ is a maximal closed left ideal. Thus by [1, p. 568, Theorem 3.4], $I^{\perp}$ is a modular maximal left ideal. Hence by the proof of [6, p. 44, Theorem 4.5] each element of I has a right adjoint and so does S.

(iii)$\Rightarrow$(iv). Let M be a nonzero left ideal of A. Since S is dense in A (see the proof of [7, p. 262, Theorem 2.1]), we have $S\cap M\neq(0)$. Therefore (iv) follows from (iii).

(iv)$\Rightarrow$(ii). Assume (iv). Let T be the set given in the proof of [**7**, p. 262, Theorem 2.1]. Then T is a dense right ideal of A and each element in T has a left adjoint. Let J be a proper closed left ideal of A. Then $J^{\perp}\neq(0)$. Let x be an element in $J^{\perp}$ which has a right adjoint x^r. Let $z\in J$ and $y\in T$. Then $(yx, z)=(x, y^l z)=0$. Since T is dense in A, we have $(Ax, z)=(0)$ and so $(A, zx^r)=(0)$ which gives $zx^r=0$. Hence $Jx^r=(0)$ and so $R(J)\neq(0)$. Therefore A is a right annihilator algebra.

(ii)$\Rightarrow$(i). Suppose (ii) holds. Let I be a minimal left ideal of A. By the argument in (ii)$\Rightarrow$(iii), $I^{\perp}$ is a modular maximal left ideal. Hence by the proof of [**7**, p. 263, Theorem 2.3], each element of I has a left adjoint and so does S. Hence by a similar argument in the proof (iv)$\Rightarrow$(ii), we can show that A is a left annihilator algebra. This completes the proof of the theorem.

REMARK. A similar form of Theorem 3.1 holds for left annihilator algebras.

THEOREM 3.2. *Let A be a modular complemented algebra such that $J\oplus J^{\perp}=A$ for all closed (left, right) ideals J of A. Then the following statements are equivalent*:

(i) *A is a dual algebra.*

(ii) *A is a left or right annihilator algebra.*

PROOF. (i)$\Rightarrow$(ii). This is clear. (ii)$\Rightarrow$(i). Assume (ii). Then by Theorem 3.1 and its remark, A is an annihilator algebra. Let I be a proper closed right ideal of A. Since $I^{\perp}\neq(0)$ and since the socle of A is dense in A, it follows that $I^{\perp}$ contains a minimal right ideal J. By the proof of Theorem 3.1, $J^{\perp}$ is a closed modular maximal right ideal and clearly $J^{\perp}\supset I$. Let $\{M_\lambda:\lambda\in\Lambda\}$ be the family of all closed modular maximal right ideals of A containing I. Then $\{M_\lambda:\lambda\in\Lambda\}$ is not empty. Let $M=\bigcap\{M_\lambda:\lambda\in\Lambda\}$. We claim that $M=I$. In fact, suppose this is not so. Then there exists an element $x\in M$ such that $x\notin I$. Since $I\oplus I^{\perp}=A$, we can write $x=x_1+x_2$ with $x_1\in I$ and $x_2\in I^{\perp}$. Hence $x_2=x-x_1\in I^{\perp}\cap M$ and so $I^{\perp}\cap M$ contains a minimal right ideal J whose orthogonal complement $J^{\perp}$ is a closed modular maximal right ideal by the above argument. Therefore by the proof of [**6**, p. 44, Theorem 4.5], J, and hence $I^{\perp}\cap M$, contains a left selfadjoint minimal idempotent e. Since $eA\subset I^{\perp}$, we have $(1-e)A=(eA)^{\perp}\supset I$. Therefore $(1-e)A\supset M$. But $e\in M$, a contradiction. Hence $M=I$. Since each $M_\lambda=R(L(M_\lambda))$, we have $I=R(L(I))$. Similarly we can show that $N=L(R(N))$ for all closed left ideals N of A. Therefore A is a dual algebra. This completes the proof.

A modular complemented algebra is said to be complete if it is complete in its inner product norm.

COROLLARY 3.3. *Let A be a complete modular complemented algebra. Then the conditions* (i), (ii), (iii) *and* (iv) *in Theorem* 3.1 *are equivalent to the statement that A is a dual algebra.*

PROOF. This follows from Theorems 3.1 and 3.2.

REMARK. It is well known that a proper H^*-algebra A is dual. This result also follows from Corollary 3.3, since every element of A has left and right adjoints.

If a modular complemented algebra is a normed algebra under its inner product norm, then we say that it is a normed modular complemented algebra.

THEOREM 3.4. *Let A be a complete normed modular complemented algebra. Then*

(i) *A is a dense left ideal of a dual B^*-algebra B.*

(ii) *If A is also a right ideal of B, then A is a dual algebra.*

PROOF. (i) For each $x \in A$, let U_x be the operation of left multiplication by x and $|U_x|$ the operator bound of U_x. Let B be the closure of $\{U_x : x \in A\}$ in the uniform topology. Since A is semisimple, the mapping $x \to U_x$ is one-one. Therefore we can identify A as a subalgebra of B. Let T be the algebraic sum of the $M^\perp$ as M ranges over the set of all closed modular maximal right ideals of A. By the proof of [**6**, p. 44, Theorem 4.5], T is dense in A, the left adjoint a^l exists for all $a \in T$ and $a^l \in T$. Clearly $U_a^l = (U_a)^*$ and so $T^* = T$ in B. Since T is dense in B, it follows that $B = B^*$. Therefore B is a B^*-algebra. Since the socle of B contains T, it follows from [**4**, p. 222, Theorem 2.1] that B is a dual algebra. For all x, $y \in A$, we have $\|xy\| \leqq |x| \, \|y\|$. This shows that A is a left ideal of B and completes the proof of (i).

(ii) Now it follows from (i) that A is a dense two-sided ideal of B. For example, a proper H^*-algebra has such a property. Let M be a nonzero right ideal of A. Then M contains a minimal idempotent e of A. It is clear that $eB = eA$ and so eA is a minimal right ideal of B. Hence by [**5**, p. 261, Lemma (4.10.1)], eA contains a hermitian minimal idempotent $f \in B$. It is easy to see that $f \in A$ and f is a left selfadjoint element in A. Since $f \in M$, it follows from Corollary 3.3 that A is a dual algebra.

4. **Hilbert algebras.** A complex algebra A is called a Hilbert algebra (see [**3**]) if it is a pre-Hilbert space with an involution $x \to x^*$ having the following properties:

(a) $(xy, z) = (y, x^*z)$ for all $x, y, z \in A$,

(b) $(x, y) = (y^*, x^*)$ for all $x, y \in A$,

(c) the operation of left multiplication $y \to xy$ is continuous on A for each $x \in A$,

(d) the set of elements of the form xy is dense in A.

It follows easily from (a) and (b) that $(yx, z) = (y, zx^*)$ for all $x, y, z \in A$. It is easy to see that the right multiplication is also continuous. Since $\|x\| = \|x^*\|$, the involution is continuous.

A net $\{e_\alpha\}$ in a normed algebra A is called an approximate identity if it is bounded and $x = \lim_\alpha xe_\alpha = \lim_\alpha e_\alpha x$ for all $x \in A$.

The following result was proved by Yood for B^*-algebras (see [7, p. 267, Theorem 3.1]).

THEOREM 4.1. *Let A be an infinite-dimensional Banach $*$-algebra with an approximate identity $\{e_\alpha\}$. Then there does not exist an inner product for A under which A becomes a pre-H^*-algebra with the same involution.*

PROOF. Suppose otherwise. Let $|x| = (x, x)^{1/2}$ be the inner product norm and let H be the completion of A in the norm $|x|$. By the proof of [7, p. 267, Theorem 3.1], H is a proper H^*-algebra and, for all $x \in A$, $|x| \leqq k\|x\|$, where k is a constant. Now it is easy to see that $\{e_\alpha\}$ is also an approximate identity in H. Since H is reflexive, by [2, p. 855, Lemma 3.8], H has an identity. Therefore by [7, p. 266, Corollary 2.7], H is finite dimensional and so is A, a contradiction. This completes the proof.

THEOREM 4.2. *Let A be a Hilbert algebra. Then A is a dual algebra if and only if $I = I^{\perp\perp}$ for all closed right ideals I of A.*

PROOF. By the proof of [7, p. 268, Lemma 3.2], $x \in xA \cap Ax$. Therefore $L(A) = R(A) = (0)$. Suppose A is a dual algebra. Let I be a closed right ideal of A. Then by the proof of [7, p. 268, Lemma 3.2], we have

$$I = R(L(I)) = ((I^\perp)^{**})^\perp = I^{\perp\perp}.$$

Conversely suppose $I = I^{\perp\perp}$ for all closed right ideals I of A. It is easy to see that $I^{\perp\perp} = R(L(I)) = I$. Let J be a closed right ideal of A. Hence $J^* = (J^*)^{\perp\perp}$ and so $J = J^{\perp\perp}$. It is now easy to see that $J = L(R(J))$. Therefore A is a dual algebra.

COROLLARY 4.3. *Let A be a Hilbert algebra. If $I \oplus I^\perp = A$ for all closed right ideals I of A, then A is a dual algebra.*

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