ON TOPOLOGICAL PROPERTIES OF SETS ADMITTING VARISOLVENT FUNCTIONS

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Abstract. Mairhuber’s theorem on Haar subspaces is generalized for the nonlinear case, where varisolvent functions are considered.

1. According to a well-known theorem of Mairhuber [4] a compact set in $\mathbb{R}^N$ is homeomorphic to a subset of a circumference $\Gamma$, if it admits a real Haar subspace with dimension $N\geq 2$. This result has been proved for general compact spaces by Curtis [2] and by Sieklucki [7]. A first extension of Mairhuber’s theorem to nonlinear families of functions was given by Dunham [3]. In this note we will derive a stronger result.

THEOREM 1. Let the compact set $Q$ admit a varisolvent family of functions. If the degree of solvency is bounded and if the maximal degree is greater than 1, then $Q$ is homeomorphic to a subset of a circumference $\Gamma$. Moreover, if the degree is an even number at some element, then the subset must be proper.

As a consequence of this theorem it is natural to consider varisolvent functions only on intervals in $\mathbb{R}$. This is the specialization used throughout the literature. The principal part of the proof will treat the case where the degree $n$ is 2. Afterwards, the induction for $n>2$ proceeds as in the proofs for linear families presented in [6], [8, p. 218].

2. We define varisolvent functions on a compact set $Q$. Let the real function $F(a, x)$ be defined for $x \in Q$ and $a \in P$, where $P$ is the parameter space [5]. For all $a \in P$ we assume $F(a, \cdot) \in C(Q)$, but there is no need to endow $P$ with a topology.

Definition. (i) $F$ has Property $Z$ of degree $m$ at $a^* \in P$, if for any $a \neq a^*$, $F(a, x) - F(a^*, x)$ has at most $m-1$ zeros for $x \in Q$.

(ii) $F$ is solvent of degree $m$ at $a^* \in P$ if given a set of $m$ distinct points $x_j \in Q, j = 1, 2, \cdots, m$, and $\varepsilon > 0$, then there exists a

\[ \delta = \delta(a^*, \varepsilon, x_1, x_2, \cdots, x_m) \]

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such that \( |F(a^*, x_j) - y_j| < \delta \) implies the existence of a parameter \( a \in P \), satisfying

\[
F(a, x_j) = y_j, \quad j = 1, 2, \ldots, m,
\]
as well as \( \|F(a, \cdot) - F(a^*, \cdot)\| < \epsilon \).

(iii) \( F \) is varisolvent if \( F \) possesses both the properties (i) and (ii) with the same degree for each \( a \in P \).

At first we prove

**Lemma 2.** Let \( \Gamma \) be a circumference and \( q \) a point outside \( \Gamma \). If \( F(a, x) \) is varisolvent on \( \Gamma \) (resp., on \( \Gamma \cup \{q\} \)), then the degree of solvence is odd (resp., at most 1) at each \( a \in P \).

**Proof.** Let \( m = m(a^*) \) be even. Select \( m \) distinct points \( x_i \in \Gamma \) in cyclic order. For sufficiently small \( \delta > 0 \) there exists a solution of

\[
F(a, x_i) = y_i, \quad i = 1, 2, \ldots, m,
\]
where \( y_i = F(a^*, x_i) + (-1)^i \delta \). Obviously, \( F(a, x) - F(a^*, x) \) has \( m \) zeros on \( \Gamma \), contradicting Property Z. This completes the proof for the statement on \( \Gamma \). Concerning \( \Gamma \cup \{q\} \), the statement for even \( m \) is a consequence of the preceding one. It is sufficient to consider odd degrees \( m = m(a^*) \). Here, select \( m - 1 \) distinct points \( x_i \in \Gamma \) in cyclic order. Define \( y_i, i = 1, 2, \ldots, m - 1 \), as above and set \( y_0 = F(a^*, q) \). Then the solution of

\[
F(a, x_i) = y_i, \quad i = 1, 2, \ldots, m - 1,
\]
\[
F(a, q) = y_0
\]
yields a contradiction to Property Z. \( \square \)

3. Before proving the main theorem we will recall the linear case. Let \( u_1, u_2 \) be a base of a two dimensional Haar subspace. Then

\[
x \rightarrow \{u_1(x), u_2(x)\}
\]
defines an injective continuous mapping of \( Q \) into the real projective line. This mapping was considered in [8, p. 221]. If \( u_2(x) \) has no zero in \( Q \), we may use the mapping \( \varphi: Q \rightarrow R \) which sends \( x \) to \( u_1(x)/u_2(x) \). We may even abandon the assumption on \( u_2(x) \). By virtue of a lemma of Schoenberg and Yang (to be mentioned below) it is only necessary to present a mapping of \( Q \setminus U \) into \( \Gamma \) for each open set \( U \subset Q \). For given \( U \) choose a function \( u_\varphi \) without zero in \( Q \setminus U \). Moreover, \( \varphi \) may be defined by the relation \( \varphi(x) \cdot u_\varphi(x) = u_1(x) \). This concept will be translated to the nonlinear case.

4. **Proof of Theorem 1.** Let \( F \) be a varisolvent function on \( Q \). We shall proceed by induction on \( n \), the number of maximal degree of solvence. Choose \( a^* \in P \) with \( m(a^*) = n \). Observe that the translation
\( \tilde{F}(a, x) = F(a, x) - F(a^*, x) \) generates a function \( \tilde{F} \), which has the same degrees of Property Z and solvence as \( F \) and which is also varisolvent. Thus we may assume \( F(a^*, x) \equiv 0 \) without loss of generality.

At first let \( n = 2 \) and let \( U \) be an arbitrary nonvoid open set in \( Q \). Select \( q_1 \in U \) and \( q_2 \in M := Q \setminus U \). From the definition of solvence we know that there exists a \( \delta > 0 \) and a mapping from the unit square in \( R^2 \) into \( C(Q) \):

\[
(y_1, y_2, q) \to F(a(y_1, y_2, q), \cdot),
\]

such that

\[
F(a(y_1, y_2, q), q_i) = \delta \cdot y_i, \quad i = 1, 2.
\]

Since \( m(a^*) \) is maximal, it follows from Property Z that the mapping is uniquely defined for fixed \( \delta \). By virtue of Theorem 1 in [1] the mapping is continuous.

For \( -1 \leq t \leq +1 \) set \( a_t = a(0, t) \). Obviously, we have \( F(a_0, \cdot) = F(a^*, \cdot) \) and \( F(a_t, q_1) = 0 \). Hence, \( m(a_t) = 2 \) and Property Z implies

\[
F(a_t, x) \neq F(a_s, x) \neq 0, \quad x \in M,
\]

for all \( s, t \in [-1, +1] \), \( s \neq t \neq 0 \). Since \( M \) is compact, it follows that

\[
\min_{x \in M} \min \{|F(a_t, x)|, |F(a_{-s}, x)|\} = \eta > 0.
\]

Furthermore, from the solvence property at \( a^* \) we conclude that there exists a parameter \( b \in P \) such that

\[
F(b, q) \neq 0 \quad \text{and} \quad \|F(b, \cdot)\| < \eta.
\]

Now we are ready to define a mapping \( \varphi : M \to [-1, +1] \) by

\[
F(a_{\varphi(x)}, x) = F(b, x), \quad x \in M.
\]

We claim that the mapping is well defined. Indeed, by virtue of (4), (5) and (6) we have

\[
F(a_t, x) \geq \eta \quad \text{and} \quad F(a_{-s}, x) \leq -\eta,
\]

or

\[
F(a_{-t}, x) \geq \eta \quad \text{and} \quad F(a_t, x) \leq -\eta,
\]

for each \( x \in M \). Hence, for each \( x \in M \) the relation \( F(a_t, x) = F(b, x) \) holds for at least one \( t \in [-1, +1] \). By virtue of (5), this value is unique. Furthermore, \( \varphi \) is injective. If \( \varphi(x_1) = \varphi(x_2) = t \), then \( F(a_t, \cdot) - F(b, \cdot) \) has two zeros, which implies \( F(a_t, \cdot) \equiv F(b, \cdot) \), contradicting \( F(b, q_1) \neq F(a_t, q_1) = 0 \). Finally, from the continuity of the mapping \([-1, +1] \times M \ni (t, x) \to F(a_t, x) - F(b, x)\), from compactness, and from the injectivity...
of ϕ, we conclude that ϕ is continuous. Hence, M = Q \ U is homeomorphic to a subset of R or a subset of Γ. Since this holds for each open set U, from Lemma 1 in [6] (the same as Lemma 2.4 in [8, p. 219]), we obtain that Q can be homeomorphically embedded into Γ, or Q is homeomorphic to a union of Γ and a single point q ∈ Y. The second case is excluded by virtue of Lemma 2. Thus, the statement for n = 2 is established.

Now, let n > 2. Assume that Theorem 1 is true for n − 1. Let U be an arbitrary open set in Q. Choose q ∈ U and a* ∈ P with m(a*) = n. Set

\[ P' = \{ a ∈ P : F(a, q) = F(a*, q) \}. \]

Obviously, F(a, x) is varisolvent for x ∈ Q \ U and a ∈ P'. By this restriction the degree of solvence is reduced by one for each a ∈ P'. Since a* ∈ P', the induction hypothesis assures that Q \ U is homeomorphic to a subset of Γ.

By the same arguments as those used for the case n = 2, the proof of the theorem is completed. □

REFERENCES


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