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ON FREE PRODUCTS OF FINITE ABELIAN GROUPS

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ABSTRACT. The purpose of this note is to show that if G is the free product of finitely many, finite abelian groups then the commutator subgroup is a finitely generated free group whose rank depends only on the number and orders of the factors. Moreover, we shall present a constructive procedure for obtaining a basis of this free group using the Kurosh rewriting process.

Let G be the free product of r free factors

$$G = *\prod_{i=1}^{r} G^i$$

and H have finite index n in G. Let d_i denote the number of double cosets HgG^i and let

$$d = \sum_{i=1}^{r} d_i.$$

By the Kurosh subgroup theorem [**1**, p. 243], H is a free product of a free group $F(H)$ and the intersections of certain conjugates of factors with H. P. J. Higgins has shown that the rank of $F(H)$ is precisely rank $(F(H))= rn-d-(n-1)$. Let A_2, G_2, G_2^i denote, respectively, the commutator subgroups of A, G, G^i. Call a group A 'commutator finite' iff A/A_2 is finite. We specialize the Higgins formula to show

THEOREM 1. *Let $G=*\prod_{i=1}^{r} G^i$ such that*
(i) *G^i is commutator finite;*
(ii) *G^i/G_2^i is of order q_i.*

Then the index of G_2 in G is $n=q_i \cdots q_r$ and $F(G_2)$ is a finitely generated free group whose rank is given by $rn-(n \sum_{i=1}^{r} (1/q_i))-(n-1)$. Moreover when the G^i are abelian the rank depends only on the number and orders of the factors.

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PROOF. We note that $G/G_2=\oplus \sum_{i=1}^r (G^i/G_2^i)$ which has order $n=q_1\cdots q_r$. The number of double coset representatives G_2gG^i is precisely the order of $(G/G_2)/(G^i/G_2^i)$ which is $n/q_i=d_i$. Thus the rank of $F(G_2)$ is derived directly by substituting in the Higgins formula. When the G^i are finite abelian it follows from [**1**, p. 249], attributed to M. Takahasi, that $G_2=F(G_2)$ is free.

We will use the Kurosh rewriting process [**1**, §4.3] to obtain the free generators for G_2 when the G^i are abelian. Choose a presentation of G^i displaying it as a direct sum of cyclic groups of prime power order:

$$G^j = \langle \cdots a_{jk} \cdots ; a_{jk}^{m_{jk}} = 1, (a_{jk}, a_{jr})\rangle,$$

where M_{jk} is a prime power $1\leqq k,\ r\leqq s_j$. Divide the generators into n classes, $\alpha_i=a_{ir}, \cdots, a_{is_i}$. We construct a regular extended Schreier system P representing the cosets of G_2 in G as follows:

(1) Let both the neutral and α_n representatives be given by

$$(*)\qquad \prod_{1\leqq j\leqq n;1\leqq k\leqq s_j} a_{jk}^{\varepsilon_{jk}} \quad \text{with} \quad 0 \leqq \varepsilon_{jk} < M_{jk}.$$

In this product, the elements a_{jk} are arranged in a lexicographic order on the subscripts j and then k.

(2) To obtain the α_i representatives we write

$$(**)\qquad \prod_{1\leqq j\leqq n; j\neq i;1\leqq k\leqq s_j} a_{jk}^{\varepsilon_{jk}} \prod_{1\leqq k\leqq s_i} a_{ik}.$$

Here each of the two products of (**) has the same ordering of the generators as in (*).

THEOREM 2. *P is a regular extended Schreier system.*

From [**1**, p. 243] and the construction of P we have that G_2 is generated by t_N, $N\in P$ and $N\neq {*N}$ where $*N$ is the neutral representative of N. The number of such t_N coincides with the rank of G_2 and so the t_N freely generate G_2.

The Kurosh rewriting process above led to a characterization of the lower central series of free products of finite direct sums of cyclic groups of order 2 [**3**] which was later generalized in [**4**].

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