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A NOTE ON THE 2/3 CONJECTURE FOR STARLIKE FUNCTIONS[1]

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ABSTRACT. Let $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular and univalent for $|z|<1$ and map $|z|<1$ onto a region which is starlike with respect to $w=0$. If r_0 denotes the radius of convexity of $w=f(z)$, $d_0=\min|f(z)|$ for $|z|=r_0$, and $d^*=\inf|\beta|$ for $f(z)\neq\beta$, then it has been conjectured that $d_0/d^*\geqq 2/3$. It is shown here that $d_0/d^*\geqq 0.380\cdots$ which improves the old estimate $d_0/d^*\geqq 0.343\cdots$. In addition an upper bound for d^* which depends on $|a_2|$ is given.

1. **Introduction.** Let S^* denote the class of functions:

$$w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

which are regular and univalent for $|z|<1$ and map $|z|<1$ onto a region which is starlike with respect to $w=0$. If r_0 denotes the radius of convexity of $w=f(z)$, $d_0=\min|f(z)|$ for $|z|=r_0$, and $d^*=\inf|\beta|$ where $f(z)\neq\beta$ for $|z|<1$, then it has been conjectured that $d_0/d^*\geqq 2/3$ [2]. Recently it has been shown that $d_0/d^*\geqq 0.343\cdots$ [3]. In this paper we will show that $d_0/d^*\geqq 0.380\cdots$. It is no loss of generality to assume that $a_2=a\geqq 0$. If this is not the case, then we replace $w=f(z)$ with $w=e^{i\theta}f(e^{-i\theta}z)$ where $\theta=\arg a_2$. This transformation does not affect the value of either d_0 or d^*.

2. **Preliminary estimates.** The following theorem improves a result appearing in [3].

THEOREM 1. *If $w=f(z)\in S^*$, then*

$$d^*\leqq \exp(-a/2) \tag{1}$$

for $0\leqq a\leqq 2$.

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PROOF. For $\alpha \in [0, 1]$ consider the function:

$$w = F(z) = z\left[\frac{f(z)}{z}\right]^{\alpha} = z + \alpha a z^2 + \sum_{n=3}^{\infty} c_n z^n. \tag{2}$$

Since:

$$zF'(z)/F(z) = (1-\alpha) + \alpha[zf'(z)/f(z)],$$

$w=F(z) \in S^*$; see [**1**, p. 221]. In [**3**], it is shown that $d_F^* \leqq 2/(2+a\alpha)$ which gives:

$$d^* = d_f^* = [d_F^*]^{1/\alpha} \leqq (2)^{1/\alpha}/(2+\alpha a)^{1/\alpha}.$$

Letting $\alpha = 1/n$ for $n=1, 2, 3, \cdots$, we obtain:

$$d^* \leqq \lim_{n\to\infty} [1 + (a/2)/n]^{-n} = \exp(-a/2).$$

The following almost immediate corollary will be useful later in this paper.

COROLLARY 1. *If* $w=f(z) \in S^*$, *then*

$$(d_0/r_0)^{1/2} \leqq \exp(-ar_0/4) \tag{3}$$

for $0 \leqq a \leqq 2$.

PROOF. Let

$$F(z) = z(f(z)/z)^{1/2} = z + \frac{a}{2} z^2 + \sum_{n=3}^{\infty} d_n z^n,$$

and

$$G(z) = r_0^{-1} F(r_0 z) = z + \frac{ar_0}{2} z^2 + \sum_{n=3}^{\infty} d_n r_0^{n-1} z^n.$$

If $\alpha = 1/2$ in (2), we see that $w=F(z) \in S^*$. Therefore, $w=G(z)$ is also in the class S^*. Therefore, by Theorem 1,

$$d_G^* = (d_0/r_0)^{1/2} \leqq \exp(-ar_0/4).$$

The following theorem enables us to obtain a lower bound for d_0/d^*.

THEOREM 2. *If* $w=f(z) \in S^*$, *then*

$$d_0/a^* \geqq 1 - (1-r_0)(d_0/r_0)^{1/2}. \tag{4}$$

PROOF. Let $z=g(w)$ denote the inverse function to $w=f(z)$. The function:

$$K(\zeta) = d^{*-1} \frac{g(d^*\zeta)}{(1 - e^{i\phi} g(d^*\zeta))^2} = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n \tag{5}$$

is regular and univalent for $|\zeta|<1$ for all $-\pi<\phi\leqq\pi$. By the classical distortion theorem we obtain:

$$|K(\zeta)| = d^{*-1}\frac{|g(d^*\zeta)|}{|(1-e^{i\phi}g(d^*\zeta))|^2} \leqq \frac{|\zeta|}{(1-|\zeta|)^2}; \tag{6}$$

see [**2**, p. 227]. Suppose $f(z_0)=w_0$ where $|z_0|=r_0$ and $|w_0|=d_0$. If we let $\zeta=w_0/d^*$, $\phi=\arg z_0$ and substitute into (6), we obtain:

$$d^{*-1}[r_0/(1-r_0)^2] \leqq (d_0/d^*)/(1-d_0/d^*)^2$$

which is equivalent to (4).

3. **Estimates for d_0/d^*.** Suppose $w=f(z)\in S^*$. It is shown in [**3**] that

$$|f(z)| \geqq |z|/(1+a|z|+|z|^2) \tag{7}$$

and

$$r_0 \geqq r_0(a) = (a+(a^2+32)^{1/2} - [2a^2+2a(a^2+32)^{1/2}+16]^{1/2})/4 \tag{8}$$

which gives the sharp estimate:

$$d_0 \geqq r_0(a)/(1+ar_0(a)+r_0^2(a)) \tag{9}$$

where equality is attained for the functions $f(z)=z(1-az+z^2)^{-1}$ for each a, $0\leqq a\leqq 2$. Inequality (8) taken with inequalities (3) and (4) gives the following:

$$d_0/d^* \geqq 1-(1-r_0(a))\exp(-ar_0(a)/4) = E_1(a) \tag{10}$$

for $0\leqq a\leqq 2$. Inequality (9) taken with inequality (1) gives the following:

$$\frac{d_0}{d^*} \geqq \frac{r_0(a)\exp(a/2)}{1+ar_0(a)+r_0^2(a)} = E_2(a) \tag{11}$$

for $0\leqq a\leqq 2$. The following two lemmas enable us to find a lower bound for the quantity d_0/d^*.

LEMMA 1. *The function $E_1(a)$ decreases for $0\leqq a\leqq 2$.*

PROOF. Let

$$H(r_0(a)) = (1-r_0(a))\exp(-ar_0(a)/4). \tag{12}$$

Since $E_1(a)=1-H(r_0(a))$, it is sufficient to prove that $H(r_0(a))$ increases for $0\leqq a\leqq 2$. It is shown in [**3**] that

$$1-ar_0(a)-6r_0^2(a)-ar_0^3(a)+r_0^4(a)=0 \tag{13}$$

which enables us to solve for a. Doing this and replacing $r_0(a)$ with R

in (12) we obtain:

$$H(R) = (1 - R)\exp[-(1/4)(1 - 6R^2 + R^4)(1 + R^2)^{-1}].$$

Furthermore, it is also shown in [3] that $r_0(a)$ is monotone decreasing for $0 \leqq a \leqq 2$. From this we see that it would be sufficient to show that $H(R)$ decreases for $r_0(2) \leqq R \leqq r_0(0)$ because the composition of two decreasing functions is an increasing function. Taking the derivative of $H(R)$ we obtain:

$$H'(R) = -\left[\frac{2 - 7R + 11R^2 + 2R^3 + R^5 - R^6}{2(1 + R^2)^2}\right] \times \exp\left[\frac{-(1 - 6R^2 + R^4)}{4(1 + R^2)}\right].$$

Therefore, to show $H(R)$ decreases it is sufficient to show

$$J(R) = 2 - 7R + 11R^2 + 2R^3 + R^5 - R^6 \geqq 0$$

for $0 \leqq R \leqq 1$ which follows because:

$$\begin{aligned} J(R) &\geqq 2 - 8R + 8R^2 + 3R^2 + 2R^3 + R^5 - R^6 \\ &\geqq 2(1 - 2R)^2 + 3R^2 + 2R^3 + R^5(1 - R) \geqq 0. \end{aligned}$$

LEMMA 2. *The function $E_2(a)$ increases for $1 \leqq a \leqq 2$.*

PROOF. If

$$F(a) = r_0(a)(1 + ar_0(a) + r_0^2(a))^{-1}, \tag{14}$$

then $E_2(a) = \exp(a/2)F(a)$ and $E_2'(a) = \exp(a/2)(F'(a) + \frac{1}{2}F(a))$. Therefore, to show $E_2(a)$ increases it is sufficient to show $F'(a) + \frac{1}{2}F(a) \geqq 0$ for $1 \leqq a \leqq 2$. Using (13) to solve for a and replacing $r_0(a)$ with R, we obtain:

$$\begin{aligned} F'(a) + \tfrac{1}{2}F(a) &= \tfrac{1}{2}[(1 + 6R^2 + R^4)(1 - R^2)^{-3}R'] \\ &\quad + [R/4(1 + R^2)(1 - R^2)^{-2}] \\ &= \frac{1 + R^2}{4(1 - R^2)^3}\left[2\,\frac{1 + 6R^2 + R^4}{(1 + R^2)}R' + R(1 - R^2)\right] \end{aligned} \tag{15}$$

where $R' = r_0'(a)$. We first note that (13) gives $R' < 0$; see [3]. Since $R = r_0(a)$ decreases with respect to a, R lies in the interval $1/4 \leqq R \leqq 1/3$. To complete the proof, it is sufficient to show:

$$W(R) = 2(1 + 6R^2 + R^4)(1 + R^2)^{-1}R' + R(1 - R^2) > 0, \tag{16}$$

for $1/4 \leqq R \leqq 1/3$. In order to show (16) holds we observe:

$$\begin{aligned} (1 + 6R^2 + R^4)(1 + R^2)^{-1} &= (1 + R^2) + 4R^2(1 + R^2)^{-1} \\ &\leqq (1 + 3^{-2}) + 4\cdot 3^{-2}(1 + 3^{-2})^{-1} < 14/9 \end{aligned}$$

and $R(1 - R^2) > (1/4)(15/16)$. Therefore we have:

$$W(R) > 2(14/9)R' + (1/4)(15/16),$$

and we see that $W(R)>0$ if $R'>-(135/1792)$. Hence, if

$$R' = R(1 + R^2)(4R^3 - 3aR^2 - 12R - a)^{-1} > -(1/14), \tag{17}$$

then we have $W(R)>0$ and the proof is complete. However, using the fact that $4R^3-3aR^2-12R-a<0$, (17) is equivalent to:

$$14R(1 + R^2) < -(4R^3 - 3aR^2 - 12R - a)$$

which is equivalent to:

$$-18R^3 + 3aR^2 - 2R + a > 0.$$

Recalling $1 \leqq a \leqq 2$, we have:

$$-18R^3 + 3R^2 - 2R + 1 > ((1/3) - R)(18R^2 + 3R + R) > 0$$

because $1/4 \leqq R \leqq 1/3$.

Using Lemmas 1 and 2 we are now ready to prove our main theorem.

THEOREM 3. $d_0/d^* \geqq 0.380 \cdots$.

PROOF. We have $E_1(A)=E_2(A)$ for $A=1.060\cdots$. From Lemma 1, $E_1(a)$ decreases for $0 \leqq a \leqq A$, and from Lemma 2, $E_2(a)$ increases for $A \leqq a \leqq 2$. Therefore, we have $d_0/d^* \geqq 0.380 \cdots$.

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