A NOTE ON THE 2/3 CONJECTURE FOR STARLIKE FUNCTIONS

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Abstract. Let $w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular and univalent for $|z| < 1$ and map $|z| < 1$ onto a region which is starlike with respect to $w = 0$. If $r_0$ denotes the radius of convexity of $w = f(z)$, $d_0 = \min \{|f(z)| : |z| = r_0\}$, and $d^* = \inf \{\beta : f(z) \neq \beta\}$, then it has been conjectured that $d_0/d^* \geq 2/3$. It is shown here that $d_0/d^* \geq 0.380 \cdots$ which improves the old estimate $d_0/d^* \geq 0.343 \cdots$. In addition an upper bound for $d^*$ which depends on $|a_2|$ is given.

1. Introduction. Let $S^*$ denote the class of functions:

$$w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are regular and univalent for $|z| < 1$ and map $|z| < 1$ onto a region which is starlike with respect to $w = 0$. If $r_0$ denotes the radius of convexity of $w = f(z)$, $d_0 = \min \{|f(z)| : |z| = r_0\}$, and $d^* = \inf \{|\beta| : f(z) \neq \beta\}$ where $f(z) \neq \beta$ for $|z| < 1$, then it has been conjectured that $d_0/d^* \geq 2/3$ [2]. Recently it has been shown that $d_0/d^* \geq 0.343 \cdots$ [3]. In this paper we will show that $d_0/d^* \geq 0.380 \cdots$. It is no loss of generality to assume that $a_2 = a \geq 0$. If this is not the case, then we replace $w = f(z)$ with $w = e^{i\theta}f(e^{-i\theta}z)$ where $\theta = \arg a_2$. This transformation does not affect the value of either $d_0$ or $d^*$.

2. Preliminary estimates. The following theorem improves a result appearing in [3].

**Theorem 1.** If $w = f(z) \in S^*$, then

$$d^* \leq \exp(-a/2)$$

for $0 \leq a \leq 2$. 

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2 Professor Tepper acknowledges support from Delaware State College.
PROOF. For \( \alpha \in [0, 1] \) consider the function:

\[
(2) \quad w = F(z) = z \left[ \frac{f(z)}{z} \right] = z + \alpha z^2 + \sum_{n=3}^{\infty} c_n z^n.
\]

Since:

\[
zF'(z)/F(z) = (1 - \alpha) + \alpha [zF'(z)/f(z)],
\]

\( w = F(z) \in S^* \); see [1, p. 221]. In [3], it is shown that \( d_F^* \leq 2/(2 + a\alpha) \) which gives:

\[
d^* = d_F^* = [d_F^*]^{1/a} \leq (2)^{1/a}/(2 + a\alpha)^{1/a}.
\]

Letting \( \alpha = 1/n \) for \( n = 1, 2, 3, \ldots \), we obtain:

\[
d^* \leq \lim_{n \to \infty} [1 + (a/2)/n]^{-n} = \exp(-a/2).
\]

The following almost immediate corollary will be useful later in this paper.

COROLLARY 1. If \( w = f(z) \in S^* \), then

\[
(3) \quad (d_0/r_0)^{1/2} \leq \exp(-ar_0/4)
\]

for \( 0 \leq a \leq 2 \).

PROOF. Let

\[
F(z) = z(f(z)/z)^{1/2} = z + \frac{a}{2} z^2 + \sum_{n=3}^{\infty} d_n z^n,
\]

and

\[
G(z) = r_0^{-1} F(r_0 z) = z + \frac{ar_0}{2} z^2 + \sum_{n=3}^{\infty} d_n r_0^{n-1} z^n.
\]

If \( \alpha = 1/2 \) in (2), we see that \( w = F(z) \in S^* \). Therefore, \( w = G(z) \) is also in the class \( S^* \). Therefore, by Theorem 1,

\[
d_G^* = (d_0/r_0)^{1/2} \leq \exp(-ar_0/4).
\]

The following theorem enables us to obtain a lower bound for \( d_0/d^* \).

THEOREM 2. If \( w = f(z) \in S^* \), then

\[
(4) \quad d_0/d^* \geq 1 - (1 - r_0)(d_0/r_0)^{1/2}.
\]

PROOF. Let \( z = g(w) \) denote the inverse function to \( w = f(z) \). The function:

\[
K(\zeta) = d_0^{* - 1} \frac{g(d_0^* \zeta)}{(1 - e^{i\phi} g(d_0^* \zeta))^2} = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n
\]
is regular and univalent for $|\zeta|<1$ for all $-\pi<\phi\leq\pi$. By the classical distortion theorem we obtain:

$$|K(\xi)| = d^{*-1} \frac{|g(d^*\xi)|}{|1 - e^{i\phi}g(d^*\xi)|^2} \leq \frac{|\zeta|}{(1 - |\zeta|^2)};$$

see [2, p. 227]. Suppose $f(z_0) = w_0$ where $|z_0| = r_0$ and $|w_0| = d_0$. If we let $\zeta = w_0/d^*$, $\phi = \arg z_0$ and substitute into (6), we obtain:

$$d^{*-1}[r_0/(1 - r_0)^2] \leq (d_0/d^*)/(1 - d_0/d^*)^2$$

which is equivalent to (4).

3. **Estimates for $d_0/d^*$**. Suppose $w = f(z) \in S^*$. It is shown in [3] that

$$|f(z)| \leq |z|/(1 + a|z| + |z|^2)$$

and

$$r_0 \geq r_0(a) = (a + (a^2 + 32)^{1/2} - \sqrt{2a^2 + 2a(a^2 + 32)^{1/2} + 16})^{1/2}$$

which gives the sharp estimate:

$$d_0 \geq r_0(a)/(1 + ar_0(a) + r_0^2(a))$$

where equality is attained for the functions $f(z) = z(1 - az + z^2)^{-1}$ for each $a$, $0 \leq a \leq 2$. Inequality (8) taken with inequalities (3) and (4) gives the following:

$$d_0/d^* \geq 1 - (1 - r_0(a))\exp(-ar_0(a)/4) = E_1(a)$$

for $0 \leq a \leq 2$. Inequality (9) taken with inequality (1) gives the following:

$$\frac{d_0}{d^*} \geq \frac{r_0(a)\exp(a/2)}{1 + ar_0(a) + r_0^2(a)} = E_2(a)$$

for $0 \leq a \leq 2$. The following two lemmas enable us to find a lower bound for the quantity $d_0/d^*$.

**Lemma 1.** The function $E_1(a)$ decreases for $0 \leq a \leq 2$.

**Proof.** Let

$$H(r_0(a)) = (1 - r_0(a))\exp(-ar_0(a)/4).$$

Since $E_1(a) = 1 - H(r_0(a))$, it is sufficient to prove that $H(r_0(a))$ increases for $0 \leq a \leq 2$. It is shown in [3] that

$$1 - ar_0(a) - 6r_0^2(a) - ar_0^2(a) + r_0^4(a) = 0$$

which enables us to solve for $a$. Doing this and replacing $r_0(a)$ with $R$
in (12) we obtain:
\[ H(R) = (1 - R) \exp[-(1/4)(1 - 6R^3 + R^4)(1 + R^5)^{-1}] \].

Furthermore, it is also shown in [3] that \( r_0(a) \) is monotone decreasing for \( 0 \leq a \leq 2 \). From this we see that it would be sufficient to show that \( H(R) \) decreases for \( r_0(2) \leq R \leq r_0(0) \) because the composition of two decreasing functions is an increasing function. Taking the derivative of \( H(R) \) we obtain:
\[ H'(R) = \frac{-(1 - 6R^2 + R^4)}{4(1 + R^2)} \times \exp\left[-(1 - 6R^2 + R^4)\right] \].

Therefore, to show \( H(R) \) decreases it is sufficient to show
\[ J(R) = 2 - 7R + 11R^3 + 2R^3 + R^5 - R^6 \geq 0 \]
for \( 0 \leq R \leq 1 \) which follows because:
\[ J(R) \geq 2 - 8R + 8R^2 + 3R^2 + 2R^3 + R^5 - R^6 \]
\[ \geq 2(1 - 2R)^2 + 3R^2 + 2R^3 + R^5(1 - R) \geq 0. \]

**Lemma 2.** The function \( E_2(a) \) increases for \( 1 \leq a \leq 2 \).

**Proof.** If
\[ F(a) = r_0(a)(1 + ar_0(a) + r_0^2(a))^{-1}, \]
then \( E_2(a) = \exp(a/2)F(a) \) and \( E_2'(a) = \exp(a/2)(F'(a) + \frac{1}{2}F(a)) \). Therefore, to show \( E_2(a) \) increases it is sufficient to show \( F'(a) + \frac{1}{2}F(a) \geq 0 \) for \( 1 \leq a \leq 2 \). Using (13) to solve for \( a \) and replacing \( r_0(a) \) with \( R \), we obtain:
\[ F'(a) + \frac{1}{2}F(a) = \frac{1}{2}[(1 + 6R^2 + R^4)(1 - R^5)^{-1}R'] + \frac{1}{2} \left[ \frac{1 + 6R^2 + R^4}{1 + R^2} \right] \frac{R'}{1 - R^5} + \frac{R(1 - R^5)}{1 - R^5} \]
\[ \frac{1 + R^2}{4(1 - R^5)^3} \left[ \frac{1 + 6R^2 + R^4}{1 + R^2} \left( R' + R(1 - R^5) \right) \right] \]
where \( R' = r_0'(a) \). We first note that (13) gives \( R' < 0 \); see [3]. Since \( R = r_0(a) \) decreases with respect to \( a \), \( R \) lies in the interval \( 1/4 \leq R \leq 1/3 \). To complete the proof, it is sufficient to show:
\[ W(R) = 2(1 + 6R^2 + R^4)(1 + R^5)^{-1}R' + R(1 - R^5) > 0, \]
for \( 1/4 \leq R \leq 1/3 \). In order to show (16) holds we observe:
\[ (1 + 6R^2 + R^4)(1 + R^5)^{-1} = (1 + R^5) + 4R^5(1 + R^5)^{-1} \leq (1 + 3^{-2}) + 4 \cdot 3^{-2}(1 + 3^{-2})^{-1} < 14/9 \]
and \( R(1 - R^2) > (1/4)(15/16) \). Therefore we have:
\[
W(R) > 2(14/9)R' + (1/4)(15/16),
\]
and we see that \( W(R) > 0 \) if \( R' > -(135/1792) \). Hence, if
\[
R' = R(1 + R^2)(4R^3 - 3aR^2 - 12R - a)^{-1} > -(1/14),
\]
then we have \( W(R) > 0 \) and the proof is complete. However, using the fact that \( 4R^3 - 3aR^2 - 12R - a < 0 \), (17) is equivalent to:
\[
14R(1 + R^2) < -(4R^3 - 3aR^2 - 12R - a)
\]
which is equivalent to:
\[
-18R^3 + 3aR^2 - 2R + a > 0.
\]
Recalling \( 1 \leq a \leq 2 \), we have:
\[
-18R^3 + 3R^2 - 2R + 1 > ((1/3) - R)(18R^3 + 3R + R) > 0
\]
because \( 1/4 \leq R \leq 1/3 \).

Using Lemmas 1 and 2 we are now ready to prove our main theorem.

**Theorem 3.** \( d_0/d^* \geq 0.380 \cdots \).

**Proof.** We have \( E_1(A) = E_2(A) \) for \( A = 1.060 \cdots \). From Lemma 1, \( E_1(a) \) decreases for \( 0 \leq a \leq A \), and from Lemma 2, \( E_2(a) \) increases for \( A \leq a \leq 2 \). Therefore, we have \( d_0/d^* \geq 0.380 \cdots \).

**Bibliography**


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