EQUIVALENT CONDITIONS FOR THE $L_p$ CONVOLUTION THEOREM ON SEMISIMPLE GROUPS

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Abstract. For certain semisimple Lie groups $G$, it is known that convolution by an $L_p(G)$ function, $1 \leq p < 2$, is a bounded operator on $L^2(G)$. This result is a consequence of the so-called "analytic continuation of the principal series" which has been carried out on these groups. However, this continuation procedure does not generalize readily to arbitrary semisimple groups. In an attempt to bypass the continuation and obtain the convolution theorem in an alternate manner, we derive in this paper several equivalent conditions for this $L_p$ convolution theorem.

1. Introduction. In recent papers ([4a], [4b]) we derived, as consequences of Kunze and Stein's analytic continuation of the principal series for $SL(n, \mathbb{C})$ [3b], several applications to harmonic analysis on these groups. Among them are two results: (i) the $L_p$ convolution theorem—the map $(f, h) \mapsto f \ast h$ is a bounded bilinear mapping of $L^p \times L^p$ into $L^2$, $1 \leq p < 2$; and (ii) an analog of the Riemann-Lebesgue Lemma for $L^p$ functions, $1 \leq p < 2$. In this paper, we shall show that these and several other conditions are actually equivalent propositions on real semisimple Lie groups.

2. Fourier analysis on semisimple groups. Let $G$ be a connected semisimple Lie group with finite center, and let $K$ be a maximal compact subgroup. Denote by $H_1, \ldots, H_r$ a complete set of Cartan subgroups of $G$ (the $H_i$ are pairwise nonconjugate, and every Cartan subgroup is conjugate to one of them). Fix $H=a$ particular $H_i$ for the moment, and set $\mathfrak{h}=\text{its Lie algebra}$. We may assume that $H$ is invariant under the Cartan involution of $G$ determined by $K$. Then $H=BA$, where $B=H \cap K$ and $A=\exp a$, $a=\text{the subalgebra of } \mathfrak{h}$ on which every root of $(g, \mathfrak{h})$ takes real values ($g$ of course is the Lie algebra of $G$). Let $P$ be a cuspidal parabolic subgroup of $G$ having $A$ as its split component. Then $P$ has a Langland's...
decomposition \( P = MAN \), where \( M \) is a (disconnected) reductive Lie group and \( B \) is a compact Cartan subgroup of \( M \). (The basic reference for the preceding is [2].)

The character group \( \hat{A} \) of \( A \) may be identified with \( \mathfrak{a}_R^* \) by \( \nu(a) = \exp iv(\log a) \), \( a \in A \), \( v \in \mathfrak{a}_R^* \). Let \( \hat{A} \) have the Euclidean topology determined by this identification. Next, let \( \tilde{M}_s \) = the unitary equivalence classes of square-integrable irreducible unitary representations of \( M \). The structure of \( \tilde{M}_s \) was computed explicitly in a recent paper [4f]. Let \( \tilde{M}_s \) have the discrete topology. (By extending the results of [4e], it is possible to show that this agrees with the relative topology induced by the hull-kernel topology on all of \( \tilde{M} \).) Set \( \Lambda = \tilde{M}_s \times \hat{A} \) with the product topology. Note that if \( H \) is compact, then \( P = G \), \( A \) is trivial and \( \Lambda = \tilde{G}_s \) is a discrete topological space.

Corresponding to each Cartan subgroup \( H_i \), we obtain a topological space \( \Lambda_i \) in the manner just described. Let \( \Lambda = \bigcup_{i=1}^r \Lambda_i \), a disjoint union. We make \( \Lambda \) into a Hausdorff topological space by declaring each \( \Lambda_i \) to be open. We denote by \( d\lambda \) the canonical Borel measure on \( \Lambda \); i.e. on each \( \Lambda_i \), \( d\lambda \) is the product of the discrete measure on \( \tilde{M}_s \) with Lebesgue measure on \( \hat{A} \).

Temporarily again, let \( H = H_t \) and \( P = MAN \). For each \( \lambda = (\sigma, v) \in \Lambda \), \( \sigma \in \tilde{M}_s \), \( v \in \hat{A} \), we form the unitary representation \( \pi^\lambda \), obtained by inducing up to \( G \) the representation \( \sigma \mapsto e^{iv(\log a)\sigma(m)} \) of \( P \). Actually these representations are “almost always” irreducible, but we shall not use that explicitly in what follows. The spaces \( \mathcal{H}^\lambda \) on which the representations \( \pi^\lambda \) act are all infinite-dimensional separable Hilbert spaces; and so they are unitarily equivalent. Henceforth, we consider them to be so identified and we denote the common space (independent of \( \lambda \) or \( H_t \)) by \( \mathcal{H} \).

Let \( \mu \) be any Borel measure on the locally compact space \( \Lambda = \bigcup_{i=1}^r \Lambda_i \). Then we may consider the Banach spaces \( L_p(\Lambda, \mathcal{H}, \mu) \), \( 1 \leq p \leq \infty \), consisting of measurable operator-valued functions \( F: \Lambda \rightarrow \text{bounded linear operators on } \mathcal{H} \) with finite norm

\[
\|F\|_p = \int_{\Lambda} \|F(\lambda)\|_p^p \, d\mu(\lambda), \quad 1 \leq p < \infty,
\]

\[
\|F\|_\infty = \text{ess sup}_{\lambda \in \Lambda} \|F(\lambda)\|_\infty.
\]

(We follow the terminology of [3a, §2] here for the most part.) In particular, we consider the Plancherel measure \( \hat{\mu} \) computed by Harish-Chandra in [2, §12]. On \( \Lambda_i \), \( d\hat{\mu}(\lambda) = C_i(\sigma, v) \, d\lambda \) where the \( C_i(\sigma, v) \) are certain continuous functions on \( \Lambda_i \). (See [2] for more extensive information on the smoothness and growth properties of these functions.) Henceforth, we denote \( L_p(\Lambda, \mathcal{H}, \hat{\mu}) \) by \( L_p(\Lambda) \).
Now define the Fourier transform \( \mathcal{F} : L_1(G) \to L_\infty(\Lambda) \) as follows:

\[
\mathcal{F}(f) = \hat{f}, \quad \hat{f}(\lambda) = \int_G f(g) \pi^*(g) \, dg, \quad f \in L_1(G).
\]

\( \mathcal{F} \) is clearly a norm-decreasing linear mapping

\[
\|f\|_1 \leq \|\hat{f}\|_\infty
\]

(2.1)

It follows from Harish-Chandra’s work that \( \mathcal{F} \) extends from \( L_1(G) \cap L_2(G) \) to an isometry of \( L_2(G) \) into the Hilbert space \( L_2(\Lambda) \):

\[
\|f\|_2 = \|\hat{f}\|_2 = \int_\Lambda \|\hat{f}(\lambda)\|^2 \, d\mu(\lambda).
\]

(2.2)

(The mapping is essentially onto. We could render it thus if, in the definition of \( L_2(\Lambda) \), we took into account the equivalences that occur amongst the \( \pi^* \).) A trivial application of [3a, Theorem 3] to (2.1) and (2.2) yields

\[
\|f\|_p = \|\hat{f}\|_q, \quad f \in L_1(G) \cap L_\infty(G),
\]

(2.3)

\( 1 < p < 2, \ 1/p + 1/q = 1 \). Hence \( \mathcal{F} \) extends to a norm-decreasing linear mapping of \( L_p(G) \) into \( L_\infty(\Lambda) \). Specifically, if \( f \in L_p(G) \), one chooses \( f_n \in L_1(G) \cap L_p(G) \), \( f_n \to f \) in \( L_p \). It follows from (2.3) that \( f_n \) is Cauchy in \( L_q(\Lambda) \). Then \( f = \mathcal{F}(f) \) is defined to be the \( L_q(\Lambda) \)-limit of the sequence \( \{f_n\} \). It is easy to check that \( f \) is independent of the choice of \( f_n \); and that if \( f \) is also in \( L_{q_1}(G) \), \( 1 \leq p_1 \leq 2 \), then the resulting \( f \) in \( L_{q_1}(\Lambda) \), \( 1/p_1 + 1/q_1 = 1 \), differs from the previous \( f \) in \( L_q(\Lambda) \) on at most a set of measure zero. Thus, given \( f \in L_p(G) \), \( 1 \leq p \leq 2 \), \( \mathcal{F}(f) = f \) is uniquely defined up to a set of measure zero.

Consider the following subsets of \( L_\infty(\Lambda) \):

\[
C(\Lambda) = \{F \in L_\infty(\Lambda) : \lambda \to \|F(\lambda)\|_\infty \text{ is a (bounded) continuous function on } \Lambda\},
\]

\[
C_\infty(\Lambda) = \{F \in C(\Lambda) : \text{the function } \lambda \to \|F(\lambda)\|_\infty \text{ vanishes at infinity}\}.
\]

Recall the latter means that for each \( \varepsilon > 0 \), the set

\[
\{\lambda \in \Lambda : \|F(\lambda)\|_\infty \geq \varepsilon\}
\]

is compact.

**Lemma.** For any \( f \in L_1(G) \), we have \( f \in C_\infty(\Lambda) \).

**Proof.** Let \( f \in L_1(G) \). Then \( f \in L_\infty(\Lambda) \). We have to show that \( \lambda \to \|\hat{f}(\lambda)\|_\infty \) is continuous and vanishes at infinity. Suppose \( f \in C_0^\infty(G) \); then by [2, p. 544] the function \( \lambda \to \text{Trace } \hat{f}(\lambda) \) has both these properties. It follows from [1, Theorem 2.3] that for \( f \in L_1(G) \), \( \lambda \to \|\hat{f}(\lambda)\|_\infty \) is a continuous function. Moreover, by [1, Corollary 1, p. 390], we get \( \lim_{\lambda \to \infty} \|\hat{f}(\lambda)\|_\infty = 0, f \in L_1(G) \). This completes the proof.
Note. In order to apply Fell’s results, we need to know that the group $C^*$-algebra of $G$ has a dense subalgebra in which every element is boundedly represented [1, p. 389]. The existence of a “large compact subgroup” in $G$ insures that this is true (see e.g. [4e, Lemma 2.2 and §6]).

3. The conditions. Let $G$ be as in §2 and $1 \leq p \leq 2$.

(i) We say that $G$ possesses the Riemann-Lebesgue property (with index $p$) if for every $f \in L_p(G)$ we have $\hat{f} \in C_\infty(\Lambda)$, i.e. there is $F \in C_\infty(\Lambda)$ such that $F = \hat{f}$ a.e.

(ii) We say that $G$ possesses the essential boundedness property (with index $p$) if for every $f \in L_p(G)$ we have $\hat{f} \in L_\infty(\Lambda)$, i.e. there is $F \in L_\infty(\Lambda)$ such that $F = \hat{f}$ a.e.

(iii) Let $U$ be a unitary representation of $G$. We say that $U$ is extendible to $L_p$ if there is a constant $A_p$ such that for every $f \in L_1(G) \cap L_p(G)$, we have $\|U_f\|_\infty \leq A_p \|f\|_p$, $U_f = \int_A f(g) U(g) \, dg$ (see [3a, p. 55]).

(iv) We say that a family $\{U^\lambda\}_{\lambda \in \Omega}$ of unitary representations of $G$ is uniformly extendible to $L_p$ if there is a single constant $A_p$ such that

$$\|U^\lambda\|_\infty \leq A_p \|f\|_p,$$

for every $f \in L_1(G) \cap L_p(G)$ and every $\lambda \in \Omega$.

(v) We say that the $L_p$ convolution theorem is valid for $G$ if there is a constant $A_p$ such that $\|f \ast h\|_2 \leq A_p \|f\|_2 \|h\|_p$ for every $f \in L_2(G)$, $h \in L_p(G)$ (see [4c, p. 765]).

4. The main result.

Theorem. Let $G$ be a connected semisimple Lie group with finite center. Let $1 \leq p \leq 2$. Then the following are equivalent:

(i) $G$ possesses the Riemann-Lebesgue property (index $p$);
(ii) $G$ possesses the essential boundedness property (index $p$);
(iii) The family $\{\pi_\lambda : \lambda \in \Lambda\}$ is uniformly extendible to $L_p$;
(iv) The regular representation of $G$ is extendible to $L_p$;
(v) The $L_p$ convolution theorem is valid for $G$.

Proof. (i)⇒(ii). This is obvious.

(ii)⇒(iii). By the hypothesis, we may consider the Fourier transform to be a linear map $\mathcal{F}: L_p(G) \to L_\infty(\Lambda)$. We show that the map has a closed graph. Suppose that the sequence $(f_n, \hat{f}_n)$ converges on the graph to $(f, F) \in L_p(G) \times L_\infty(\Lambda)$. Then

$$\|f_n - f\|_p \to 0$$

and

$$\|\hat{f}_n - F\|_\infty \to 0.$$
Applying (2.3) to (4.1) we obtain \( \| f_n - \hat{f} \|_q \to 0 \). That is,
\[
\int_\Lambda \| f_n(\lambda) - \hat{f}(\lambda) \|_q^q \, d\mu(\lambda) \to 0.
\]
Therefore, we may choose a subsequence \( f_{n_j} \) such that \( \| f_{n_j}(\lambda) - \hat{f}(\lambda) \|_q \to 0 \) a.e. But then
\[
\| f_{n_j}(\lambda) - \hat{f}(\lambda) \|_\infty \leq \| f_{n_j}(\lambda) - \hat{f}(\lambda) \|_q \to 0 \quad \text{a.e.}
\]
Finally, (4.2) implies that
\[
\text{ess sup } \| f(\lambda) - F(\lambda) \|_\infty \to 0.
\]
It follows from (4.3) and (4.4) that \( F = \hat{f} \) a.e. This is precisely what is needed in order to demonstrate that \( \mathcal{F} \) has a closed graph.

By the closed graph theorem, \( \mathcal{F} : L_p(G) \to L_\infty(\Lambda) \) is a bounded linear map, i.e. there is \( A_p \) such that
\[
\text{ess sup } \| \hat{f}(\lambda) \|_\infty \leq A_p \| f \|_p, \quad f \in L_p(G).
\]
Hence, by the Lemma,
\[
\| \hat{f}(\lambda) \|_\infty \leq A_p \| f \|_p, \quad f \in L_1(G) \cap L_p(G),
\]
for all \( \lambda \in \Lambda \).

(iii) \( \Rightarrow \) (v). This proof is fairly standard by now (see e.g. [3a, Theorem 9] or [4a, Theorem 4]). It is enough to consider \( f \in L_1(G) \cap L_2(G), h \in L_1(G) \cap L_p(G) \). Then
\[
\| f \ast h \|_2^2 = \int_\Lambda \| (f \ast h)^{\wedge}(\lambda) \|_2^2 \, d\hat{\mu}(\lambda) = \int_\Lambda \| \hat{f}(\lambda) \|_2^2 \, d\hat{\mu}(\lambda)
\]
\[
\leq \int_\Lambda \| \hat{f}(\lambda) \|_\infty \| \hat{h}(\lambda) \|_2 \, d\hat{\mu}(\lambda) \leq A_p^2 \| h \|_2^2 \int_\Lambda \| \hat{f}(\lambda) \|_\infty \, d\hat{\mu}(\lambda)
\]
\[
= A_p^2 \| h \|_2^2 \| f \|_p^2.
\]

(v) \( \Rightarrow \) (iv). Kunze and Stein have already observed [3a, p. 55] that these two statements are in fact different ways of saying the same thing.

(iv) \( \Rightarrow \) (i). It is clear from the Plancherel formula (2.2) that the Hilbert space \( L_2(G) \) decomposes into the direct integral \( \int_\Lambda H^2 \, d\hat{\mu}(\lambda) \) via the Fourier transform. Moreover, the components of the regular representation under this decomposition are precisely \( \pi^\lambda, \lambda \in \Lambda \). Therefore the second part of the proof of [3a, Theorem 11] shows that there is \( A_p \) such that
\[
\text{ess sup } \| \hat{f}(\lambda) \|_\infty \leq A_p \| f \|_p, \quad f \in L_1(G) \cap L_p(G).
\]
But by continuity (i.e. by the Lemma), we obtain

\[(4.5) \quad \|f(\lambda)\|_\infty \leq A_\lambda \|f\|_\|, \quad f \in L_1(G) \cap L_p(G), \lambda \in \Lambda.\]

Let \(\lambda \in \Lambda\) be arbitrary but fixed and suppose \(f \in L_p(G)\). Choose \(f_n \in L_1(G) \cap L_p(G)\), \(f_n \to f\) in \(L_p\). Then \(\{f_n(\lambda)\}\) is a Cauchy sequence in the space \(\mathcal{B}\) of all bounded linear operators on \(\mathcal{H}\) (because of (4.5)). Let \(\hat{f}(\lambda)\) be its limit. It is routine to see (by a proof analogous to the argument used in (ii) \(\Rightarrow\) (iii)) that for any \(f \in L_p(G)\), \(\hat{f} = f\) a.e. Thus \(\hat{f} \in L_\infty(\Lambda)\) and we have a stronger version of (4.5), namely

\[(4.6) \quad \|\hat{f}(\lambda)\|_\infty \leq A_\lambda \|f\|_\|, \quad f \in L_p(G), \lambda \in \Lambda.\]

It only remains to show that \(\hat{f} \in C_\infty(\Lambda)\).

Let \(f \in L_p(G), \varepsilon > 0\). Choose \(h \in L_1(G) \cap L_p(G)\) such that \(\|f-h\|_\| < \varepsilon/2A_\lambda\).

Fix an arbitrary \(\lambda_0 \in \Lambda\). We show continuity at \(\lambda_0\). If \(\lambda\) is sufficiently close to \(\lambda_0\), then

\[(4.7) \quad |\|\hat{h}(\lambda)\|_\infty - \|\hat{h}(\lambda_0)\|_\infty| < \varepsilon.\]

Moreover, using (4.6) and (4.7), we have

\[
\begin{align*}
\|f(\lambda)\|_\infty - \|f(\lambda_0)\|_\infty & \leq |\|f(\lambda)\|_\infty - \|\hat{f}(\lambda)\|_\infty| \\
& \quad + |\|\hat{h}(\lambda)\|_\infty - \|\hat{h}(\lambda_0)\|_\infty| \\
& \quad + |\|\hat{f}(\lambda_0)\|_\infty - \|\hat{f}(\lambda_0)\|_\infty| \\
& < 2 \sup_{\lambda} \|\hat{f}(\lambda) - \hat{h}(\lambda)\|_\infty + \varepsilon \leq 2\varepsilon.
\end{align*}
\]

That is, \(\|f(\lambda)\|_\infty\) is close to \(\|f(\lambda_0)\|_\infty\) and we have proven continuity.

Finally, since \(\|\hat{f}(\lambda) - \hat{h}(\lambda)\|_\infty < \varepsilon/2\) for all \(\lambda \in \Lambda\), we have

\[(4.8) \quad \{\lambda \in \Lambda: \|\hat{f}(\lambda)\|_\infty \geq \varepsilon\} \subseteq \{\lambda \in \Lambda: \|\hat{h}(\lambda)\|_\infty \geq \varepsilon/2\}.\]

But \(h \in L_1(G)\) implies (by the Lemma) that the right side of (4.8) is compact. Since \(\lambda \to \|\hat{f}(\lambda)\|_\infty\) is continuous, the left side is closed, and therefore also compact. Because \(\varepsilon\) is arbitrary, we conclude that \(\hat{f} \in C_\infty(\Lambda)\).

5. Remarks. (i) If \(G\) is compact, it is easy to see that the conditions of the Theorem are fulfilled for any \(p \geq 1\) (since \(L_p(G) \subseteq L_1(G)\)). However, if \(G\) is noncompact, the \(L_2\) convolution theorem is false (see [5]). For certain groups, these five conditions are true when \(1 \leq p < 2\), and it has been conjectured that they hold for any \(G\) as in §2 (see [4d, §1]). This question is still open.

(ii) The specific groups for which the \(L_p\) convolution theorem has been substantiated have all been handled in essentially the same way; namely by an analytic continuation of the representations \(\pi_\lambda, \lambda = (\sigma, \nu)\), to uniformly bounded representations depending on \(\nu \in \alpha^c\). This method has
only borne fruit when applied to a minimal parabolic subgroup—the case of nonminimal parabolics appears to be significantly more difficult. One is naturally led to the question: Can we obtain the $L_p$ convolution theorem by some means other than analytic continuation? This was one of our main motivations in looking for equivalent conditions.

(iii) The set $C_\alpha(A)$ is not a linear space (because the continuity of $\lambda \rightarrow \|F_\lambda\|_\infty$ and $\lambda \rightarrow \|F_\lambda(\alpha)\|_\infty$ does not imply the continuity of $\lambda \rightarrow \|F_\lambda(\alpha) + F_\lambda(\beta)\|_\infty$). However, in those groups for which analytic continuation has been carried out, it is possible to show that for $f \in L_1(G)$, $\tilde{f}(\lambda)$ is continuous in the uniform topology; i.e.

$$\lim_{\lambda \to \lambda_0} \|\tilde{f}(\lambda) - \tilde{f}(\lambda_0)\|_\infty = 0.$$ 

It seems likely then that one can replace $C_\infty(A)$ in the Theorem by the linear space $C_{\infty}(A) = \{F \in L_\infty(A): \lambda \rightarrow F(\lambda)\}$ is continuous in the uniform topology and $\lim_{\lambda \to \infty} \|F(\lambda)\|_\infty = 0\}.$

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