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PERMUTABLE PRONORMAL SUBGROUPS

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ABSTRACT. Let G be a finite solvable group. It is shown that a certain class of pronormal subgroups reducing a given Sylow system is a lattice of permutable subgroups.

All groups considered here are finite solvable. A subgroup V of a group G is said to be *p-normally embedded* in G if a Sylow p-subgroup P of V is also a Sylow subgroup of P^G. A subgroup V of G is said to be *normally embedded* in G if it is p-normally embedded for every prime p. A normally embedded subgroup is necessarily pronormal [**1**, Theorem 2.3]. We prove the following two theorems:

THEOREM 1. *A p-subgroup P of G is a Sylow subgroup of P^G if it is a Sylow subgroup of $\langle P, P^x\rangle$ for every $x \in G$.*

THEOREM 2. *Let Σ be a Sylow system of the group G. Then the set of normally embedded subgroups reducing Σ is a lattice of permutable subgroups.*

A Sylow system [**2**] is a complete set of permutable Hall subgroups. A Sylow system Σ of G is said to *reduce* into a subgroup U if $\{U \cap H | H \in \Sigma\}$ is a Sylow system of U.

PROOF OF THEOREM 1. We proceed by induction on the order $|G|$. Let A be a minimal normal subgroup of G. Since PA/A satisfies the hypothesis in G/A, PA/A is a Sylow p-subgroup of $(PA)^G/A = P^GA/A$. Therefore we may assume that $O_{p'}(G)=1$ and $\mathrm{core}(P)=1$. Then PA is a Sylow subgroup of $H=P^GA$.

Case 1. $H \neq G$. Then P is a Sylow subgroup of $N=P^H$. Since P is a pronormal subgroup of G and a subnormal subgroup of $N_G(PA)$, $N_G(P) \supseteq N_G(PA)$. Then $G=H\cdot N_G(PA)=H\cdot N_G(P)$ and $\mathrm{core}(N)=\bigcap_{x\in G} N^x = \bigcap_{x\in N_G(P)} N^x \supseteq P$. Hence P is a Sylow subgroup of a normal subgroup of G.

Case 2. $H=G$. Since P is a Sylow subgroup of $\langle P, P^x\rangle$, $A \cap \langle P, P^x\rangle = A \cap P \cap P^x$. Therefore $A \cap P = A \cap \mathrm{core}(P) = 1$. Hence A lies in the center of

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PA and, by a theorem of Gaschütz [**4**, Hauptsatz 17.4, p. 121], A is complemented by a subgroup K. Let $Q=PA\cap K$. Since A lies in the center of PA, $\mathrm{core}(K)=\bigcap_{a\in A} K^a \supseteq Q$. Then $A\cdot\mathrm{core}(K)\supseteq (PA)^G=G$ so that $K=\mathrm{core}(K)$ and A lies in the center of G. Let B be a minimal normal subgroup of G contained in K. Similarly, B is a p-subgroup lying in the center of G. Then $\mathrm{core}(P)\supseteq AB\cap P\neq 1$. This proves the theorem.

PROOF OF THEOREM 2. (1) Let U, V be normally embedded permutable subgroups of G. Then UV and $U\cap V$ are normally embedded. By a theorem of Wielandt [**4**, Satz, 4.6, p. 676], there are Sylow p-subgroups P, Q of U, V respectively such that PQ is a Sylow subgroup of UV. Since $|UV|/|PQ|=(|U|/|P|)(|V|/|Q|)(|P\cap Q|/|U\cap V|)$ is a p'-number, $P\cap Q$ is a Sylow subgroup of $U\cap V$. On the other hand, $P\cap Q=P\cap(PQ\cap Q^G)=P\cap(P^G\cap Q^G)$ is a Sylow subgroup of $P^G\cap Q^G$. Thus $U\cap V$ is p-normally embedded.

(2) If a Sylow system Σ of G reduces into permutable subgroups U and V then Σ reduces into $U\cap V$ and UV. That Σ reduces into $U\cap V$ is due to Shamash as quoted in [**5**, Lemma 2, p. 230]. Let P be a Sylow p-subgroup of G belonging to Σ. Then $P\cap U$ and $P\cap U$ are Sylow subgroups of U and V respectively. Since Σ reduces to $U\cap V$, $|U\cap V|/|P\cap U\cap V|$ is a p'-number. It follows that

$$|UV|/|(P\cap U)(P\cap V)| = (|U|/|P\cap U|)(|V|/|P\cap V|)(|P\cap U\cap V|/|U\cap V|)$$

is a p'-number. Hence $(P\cap U)(P\cap V)=P\cap UV$ is a Sylow p-subgroup of UV.

(3) Let U, V be normally embedded subgroups of the group G. If U and V reduce a Sylow system Σ, then U and V are permutable. We use induction on the order $|G|$. Let A be a minimal normal subgroup of G. Then UA/A and VA/A permute. Then U and VA permute. Since Σ reduces into UVA, $G=UVA$. Let p be the prime dividing $|A|$. Suppose that G has a normal p'-subgroup B. Then U and VB permute and $G=UVB$. Let P be the Sylow p-subgroup of G belong to Σ. Then $P=(P\cap U)(P\cap VB)=(P\cap U)(P\cap V)\subseteq UV$, so that $G=UV$. Therefore we may assume that $O_{p'}(G)=1$. If p does not divide $|UV|$, then UV is the Sylow p-complement of G belonging to Σ. Therefore we may assume that p divides the order $|U|$. Then $O_p(G)\cap(P\cap U)^G$ is a nonidentity normal subgroup of G contained in $P\cap U$. Replace A by $O_p(G)\cap(P\cap U)^G$; we get $G=UV$.

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