PERMUTABLE PRONORMAL SUBGROUPS

TI YEN

Abstract. Let G be a finite solvable group. It is shown that a certain class of pronormal subgroups reducing a given Sylow system is a lattice of permutable subgroups.

All groups considered here are finite solvable. A subgroup V of a group G is said to be p-normally embedded in G if a Sylow p-subgroup P of V is also a Sylow subgroup of $P^G$. A subgroup V of G is said to be normally embedded in G if it is p-normally embedded for every prime p. A normally embedded subgroup is necessarily pronormal [1, Theorem 2.3]. We prove the following two theorems:

Theorem 1. A p-subgroup P of G is a Sylow subgroup of $P^G$ if it is a Sylow subgroup of $\langle P, P^x \rangle$ for every $x \in G$.

Theorem 2. Let $\Sigma$ be a Sylow system of the group G. Then the set of normally embedded subgroups reducing $\Sigma$ is a lattice of permutable subgroups.

A Sylow system [2] is a complete set of permutable Hall subgroups. A Sylow system $\Sigma$ of G is said to reduce into a subgroup U if $\{U \cap H | H \in \Sigma\}$ is a Sylow system of U.

Proof of Theorem 1. We proceed by induction on the order $|G|$. Let $A$ be a minimal normal subgroup of G. Since $PA/A$ satisfies the hypothesis in $G/A$, $PA/A$ is a Sylow p-subgroup of $(PA)^G/A = P^G A/A$. Therefore we may assume that $O_p(G) = 1$ and $core(P) = 1$. Then $PA$ is a Sylow subgroup of $H = P^G A$.

Case 1. $H \neq G$. Then $P$ is a Sylow subgroup of $N = P^H$. Since $P$ is a pronormal subgroup of $G$ and a subnormal subgroup of $N_G(PA)$, $N_G(P) \supseteq N_G(PA)$. Then $G = H \cdot N_G(PA) = H \cdot N_G(P)$ and $core(N) = \bigcap_{x \in G} N^x = \bigcap_{x \in N_G(P)} N^x \supseteq P$. Hence $P$ is a Sylow subgroup of a normal subgroup of G.

Case 2. $H = G$. Since $P$ is a Sylow subgroup of $\langle P, P^x \rangle$, $A \cap \langle P, P^x \rangle = A \cap P \cap P^x$. Therefore $A \cap P = A \cap core(P) = 1$. Hence $A$ lies in the center of

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PA and, by a theorem of Gaschütz [4, Hauptsatz 17.4, p. 121], A is complemented by a subgroup K. Let $Q = PA \cap K$. Since A lies in the center of PA, $\text{core}(K) = \bigcap_{a \in A} K = Q$. Then $A \cdot \text{core}(K) \subseteq (PA)^G = G$ so that $K = \text{core}(K)$ and A lies in the center of G. Let B be a minimal normal subgroup of G contained in K. Similarly, $B$ is a $p$-subgroup lying in the center of $G$. Then $\text{core}(P) \supseteq AB \cap P \neq 1$. This proves the theorem.

**Proof of Theorem 2.** (1) Let $U, V$ be normally embedded permutable subgroups of $G$. Then $UV$ and $U \cap V$ are normally embedded. By a theorem of Wielandt [4, Satz, 4.6, p. 676], there are Sylow $p$-subgroups $P, Q$ of $U, V$ respectively such that $PQ$ is a Sylow subgroup of $UV$. Since $|UV|/|PQ| = (|U|/|P|)(|V|/|Q|)(|P \cap Q|/|U \cap V|)$ is a $p'$-number, $P \cap Q$ is a Sylow subgroup of $U \cap V$. On the other hand, $P \cap Q = P \cap (PQ \cap Q^G) = P \cap (P^G \cap Q^G)$ is a Sylow subgroup of $P^G \cap Q^G$. Thus $U \cap V$ is $p$-normally embedded.

(2) If a Sylow system $\Sigma$ of $G$ reduces to permutable subgroups $U$ and $V$ then $\Sigma$ reduces into $U \cap V$ and $UV$. That $\Sigma$ reduces into $U \cap V$ is due to Shamash as quoted in [5, Lemma 2, p. 230]. Let $P$ be a Sylow $p$-subgroup of $G$ belonging to $\Sigma$. Then $P \cap U$ and $P \cap V$ are Sylow subgroups of $U$ and $V$ respectively. Since $\Sigma$ reduces to $U \cap V$, $|U \cap V|/|P \cap U \cap V|$ is a $p'$-number. It follows that

$$|UV|/|P \cap U \cap V|$$

is a $p'$-number. Hence $(P \cap U)(P \cap V) = P \cap UV$ is a Sylow $p$-subgroup of $U$. V.

(3) Let $U, V$ be normally embedded subgroups of the group $G$. If $U$ and $V$ reduce a Sylow system $\Sigma$, then $U$ and $V$ are permutable. We use induction on the order $|G|$. Let $A$ be a minimal normal subgroup of $G$. Then $UA/A$ and $VA/A$ permute. Then $U$ and $V$ permute. Since $\Sigma$ reduces into $UVA$, $G = UVA$. Let $p$ be the prime dividing $|A|$. Suppose that $G$ has a normal $p'$-subgroup $B$. Then $U$ and $VB$ permute and $G = UBV$. Let $P$ be the Sylow $p$-subgroup of $G$ belong to $\Sigma$. Then $P = (P \cap U)(P \cap VB) = (P \cap U)(P \cap V) \subseteq UV$, so that $G = UV$. Therefore we may assume that $O_p'(G) = 1$. If $p$ does not divide $|UV|$, then $UV$ is the Sylow $p$-complement of $G$ belonging to $\Sigma$. Therefore we may assume that $p$ divides the order $|U|$. Then $O_p(G) \cap (P \cap U)^G$ is a nonidentity normal subgroup of $G$ contained in $P \cap U$. Replace $A$ by $O_p(G) \cap (P \cap U)^G$; we get $G = UV$.

**References**


Department of Mathematics, Michigan State University, East Lansing, Michigan 48823