A CHARACTERIZATION OF STRONGLY MEASURABLE PETTIS INTEGRABLE FUNCTIONS

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Abstract. Theorem. Let $X$ be a Banach space and $(\Omega, \Sigma, \mu)$ be a finite measure space. A strongly measurable $f: \Omega \to X$ is Pettis integrable if and only if there exists a Young’s function $\Phi$ with $\lim_{t \to \infty} \Phi(t)/t = \infty$ such that $x^*f \in L^\Phi(\mu)$ for all $x^* \in X^*$.

In 1969, Brooks [1] gave a characterization of strongly measurable Pettis integrable functions by use of the Orlicz-Pettis theorem. The purpose of this note is to give another characterization in terms of Orlicz spaces. Throughout this note $X$ is a Banach space with continuous dual $X^*$. $(\Omega, \Sigma, \mu)$ is a fixed finite measure space. $f: \Omega \to X$ is called strongly measurable if it is the $\mu$-almost everywhere limit of simple functions of the form $\sum_{i=1}^n x_i \chi_{E_i}$ where $x_i \in X$, $E_i \in \Sigma$ and $\chi_{E_i}$ is the characteristic or indicator function of $E_i$. A strongly measurable $f: \Omega \to X$ is called Pettis integrable if $x^*f \in L^1(\mu)$ for all $x^* \in X^*$ and for each $E \in \Sigma$ there is $x_E \in X$ satisfying $x^*(x_E) = \int_E x^*f \, d\mu$ for all $x^* \in X^*$. A Young’s function is an increasing convex function $\Phi$ defined on $[0, \infty)$ such that $\Phi(0) = 0$. The Orlicz space $L^\Phi(\mu)$ is the space of all (equivalence classes of) functions $f$ such that $\int_{\Omega} \Phi(|f|/k) \, d\mu < \infty$ for some $k > 0$.

The following theorem gives the promised characterization of Pettis integrable strongly measurable functions. In a sense this theorem is an analogue of LaVallée Poussin’s theorem for vector valued functions.

Theorem. Let $f: \Omega \to X$ be strongly measurable, $f$ is Pettis integrable with respect to $\mu$ if and only if there exists a Young’s function $\Phi$ such that

(i) $\lim_{t \to \infty} \Phi(t)/t = + \infty$, and

(ii) $x^*f \in L^\Phi(\mu)$ for each $x^* \in X^*$.

Proof (Sufficiency). Suppose $x^*f \in L^\Phi(\mu)$ for all $x^* \in X^*$. Then if $T: X^* \to L^\Phi(\mu)$ is defined by $T(x^*) = x^*f$, $T$ is linear. Furthermore $T$ is closed since if $x_n^* \to x^*$ in $X^*$ and $T(x_n^*) \to g$ in $L^\Phi(\mu)$ norm, then $x^*f = g$ a.e. $[\mu]$ because $x_n^*f \to x^*f = g$ pointwise almost everywhere. According to Banach’s closed graph theorem, $T$ is continuous. Hence if $g \in L^\Psi$ where $\Psi$ is

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complementary to $\Phi$, and $S(g)$ is defined on $\mathcal{X}^*$ by
\[
S(g)(x^*) = \int_{\Omega} g x^* f \, d\mu, \quad x^* \in \mathcal{X}^*,
\]
then $S$ is linear in $g$. Moreover for $x^* \in \mathcal{X}^*$,
\[
|S(g)(x^*)| = \left| \int_{\Omega} g x^* f \, d\mu \right| \leq \|g\|_V \|x^* f\|_\Phi \leq \|g\|_V \|x^*\| \|T\|.
\]
Hence $S(g) \in \mathcal{X}^{**}$ and $S(g)$ is the “Gelfand” integral of $gf$.

Next we shall show that $S$ actually maps bounded functions in $L^\Psi$ into the closed subspace $\Phi$ of $\mathcal{X}^{**}$. To this end, set $E_n = \{\omega: \|f(\omega)\| \leq n\}$. If $g$ is supported in one of the $E_n$’s, say $E_{n_0}$, consider
\[
S(g)(x^*) = \int_{\Omega} g x^* f \, d\mu, \quad x^* \in \mathcal{X}^*.
\]
Since $f$ is bounded on $E_{n_0}$, $gf$ is actually Bochner integrable. Hence
\[
S(g)(x^*) = x^* \left( \text{Bochner} - \int_{\Omega} gf \, d\mu \right);
\]
i.e. $S(g) = \text{Bochner} - \int_{\Omega} gf \, d\mu$ for $g$ supported inside one of the $E_n$’s. Now condition (i) implies $\Psi(t) < \infty$ when $t < \infty$. Hence by [4, Theorem 10.3, p. 87], $\lim_{\mu(E_n) \to 0} \|g\chi_{E_n}\|_\Psi = 0$ for all $g$ in the closed subspace of $L^\Psi$ determined by the simple functions. Since $E_n \uparrow \Omega$, the continuity of $S$ implies $S(g) = \lim_n S(g\chi_{E_n})$ strongly in $\mathcal{X}^{**}$ for all $g$ in the closed subspace of $L^\Psi$ determined by the simple functions. In particular, $S(\chi_E) = \lim_n S(\chi_{E \cap E_n}) = \lim_n \text{Bochner} - \int_{\Omega} \chi_{E \cap E_n} f \, d\mu \in \mathcal{X}$. Hence $S(\chi_E)(x^*) = \int_{\Omega} \chi_{E \cap E_n} f \, d\mu$ and $f$ is Pettis integrable.

(Necessity). Suppose $f$ is Pettis integrable. By a theorem of Pettis [6, Corollary 2.51, p. 284], $\lim_{\mu(E_n) \to 0} \int_{E_n} |x^* f| \, d\mu = 0$ uniformly for $x^*$ in the unit ball of $\mathcal{X}^*$. Also $\sup_{\|x^*\| \leq 1} \|x^* f\|_{L^1} < \infty$. An appeal to LaVallée Poussin’s theorem [5, Theorem 22, p. 19] establishes the existence of a convex function $\Phi$ defined on $[0, \infty)$ which is nonnegative, increasing, convex and satisfying $\lim_{t \to \infty} \Phi(t)/t = +\infty$ such that
\[
\sup_{\|x^*\| \leq 1} \int_{\Omega} \Phi(|x^* f|) \, d\mu < \infty.
\]
By setting $\Phi_1(t) = \Phi(t) - \Phi(0)$, one finds that $\Phi_1$ is a Young’s function satisfying (i) and $x^* f \in L^{\Phi_1}(\mu)$ for all $x^* \in \mathcal{X}^*$. This completes the proof of the theorem.
REFERENCES


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