

LINEAR PROCESSES GENERATED BY INDEPENDENT RANDOM VARIABLES

W. A. WOYCZYŃSKI¹

ABSTRACT. In a recent paper R. Dudley gave a characterization of those sequences of independent and identically distributed random variables which are l_p -compatible for $p \geq 1$. In the present note we extend his result into $p \in (0, 1]$ and provide some conditions (necessary or sufficient) for l_φ -compatibility of a sequence of independent random variables not necessarily identically distributed.

Let $\{X_n\} \subset \mathcal{M}(\Omega, \mathcal{F}, P)$ be a sequence of random variables. If for every element (a_n) of a complete metric linear space \mathcal{X} of real sequences in which coordinate functionals $(a_n) \rightarrow a_m, m=1, 2, \dots$, are continuous, the series $\sum a_n X_n$ converges in probability (we shall say in this case that $\{X_n\}$ is \mathcal{X} -compatible), then the mapping

$$X: \mathcal{X} \ni (a_n) \rightarrow \sum a_n X_n \in \mathcal{M}(\Omega, \mathcal{F}, P)$$

defines a linear continuous stochastic process on \mathcal{X} (for basic definitions the reader is referred to [1], [4] and references given therein). The linearity is clear and the continuity follows from the classical Banach theorem which says that every Borel homomorphism of a complete metric group into a metric is continuous and from a remark that X is a pointwise limit of continuous (because finite-dimensional) hence Borel functions $\sum_{n=1}^{n_0} a_n X_n$. R. Dudley [2] gave a characterization of those sequences of independent and identically distributed random variables which are l_p -compatible for $p \geq 1$. In this note we extend his result onto $p \in (0, 1]$ and provide a necessary condition for the sequence of independent random variables (not necessarily identically distributed) to be l_φ -compatible and the sufficient condition for the same sequence to be l_p -compatible for $p > 0$. However it seems that they are rather far from the conditions which are necessary and sufficient simultaneously.

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Now let $\{X_n\}$ be a sequence of independent random variables with distribution functions F_n and let $\varphi, \psi: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ be nondecreasing continuous functions vanishing only at the origin. Call the sequence $\{X_n\}$ (or $\{F_n\}$) lower [upper] weakly φ -finite if, for some $C > 0$,

$$\liminf_n \left[\sup_n \right] P\{|X_n| \geq M\} \leq C/\varphi(M), \quad 0 \leq M < \infty.$$

We shall say that φ satisfies the Δ_2 -condition ($\varphi \in \Delta_2$) if for some positive constant K , $\varphi(2t) \leq K\varphi(t)$, $0 \leq t < \infty$. If $\varphi \in \Delta_2$ then the sequence space

$$l_\varphi \equiv \{(a_n) \in \mathbf{R}^\infty: \sum \varphi(|a_n|) < \infty\}$$

is a linear space which can be equipped with a nonhomogeneous norm (e.g. $\|(a_n)\|_\varphi = \inf\{c: c > 0, \sum \varphi(|a_n/c|) \leq c\}$) which makes l_φ a complete metric linear space with continuous coordinate functionals (see e.g. [3, Theorems 1.4, 1.62 and 1.82]).

The propositions given below explain the relationship between lower [upper] finiteness and the existence of moments for a sequence $\{X_n\}$.

PROPOSITION 1. *If, for some subsequence $\{n_k\}$, $k=1, 2, \dots$, $E\varphi(|X_{n_k}|)$ are finite and $\liminf_k E\varphi(|X_{n_k}|) = L < \infty$, then the sequence $\{X_n\}$ is lower weakly φ -finite.*

PROOF.

$$\begin{aligned} \liminf_n P\{|X_n| \geq M\} \cdot \varphi(M) &\leq \liminf_k P\{|X_{n_k}| \geq M\} \cdot \varphi(M) \\ &\leq \liminf_k \int_{|X_{n_k}| \geq M} \varphi(|X_{n_k}|) dP \leq \liminf_k E\varphi(|X_{n_k}|) = L < \infty. \end{aligned}$$

PROPOSITION 2. *Let $\varphi \in \Delta_2$, φ, ψ be differentiable with monotone φ' , and let $\psi(t)\varphi^{-1}(t) \rightarrow 0$ as $t \rightarrow +\infty$ in such a way that $\psi(t)\varphi^{-1}(t)t^{-1} \in L^1(\alpha, \infty)$ for some $\alpha > 0$. In those conditions if $\varphi(M) \limsup_n P\{|X_n| \geq M\} \leq C$, $0 \leq M < \infty$ (even more so if $\{X_n\}$ is upper weakly φ -finite) then $\limsup_n E\psi(|X_n|) < \infty$.*

PROOF. Note that

$$\begin{aligned} \limsup_n E\psi(|X_n|) &= \limsup_n \int_{-\infty}^{\alpha} \psi(|t|) dF_n(t) \\ &\quad + \limsup_n \int_{|X_n| \leq \alpha} \psi(|X_n|) dP + \limsup_n \int_{\alpha}^{\infty} \psi(|t|) d(F_n(t) - 1). \end{aligned}$$

The second term being bounded by $\psi(\alpha)$ it is sufficient to evaluate the first and the third ones. We restrict ourselves to the evaluation of the third term (the procedure for the first one is analogous). Integrating by parts we

get that

$$\begin{aligned} \limsup_n \int_\alpha^\infty \psi(t) d(F_n(t) - 1) \\ = \limsup_n \left[\frac{\psi(t)}{\varphi(t)} \varphi(t)(F_n(t) - 1) \right]_\alpha^\infty + \int_\alpha^\infty \psi'(t)(1 - F_n(t)) dt. \end{aligned}$$

By our assumptions, $\psi(t) \cdot \varphi^{-1}(t) \rightarrow 0$ as $t \rightarrow \infty$ and $\limsup_n \varphi(t)(F_n(t) - 1)$ remains bounded. Hence the first summand is bounded by $\psi(\alpha)$. As to the second one we have that

$$\limsup_n \int_\alpha^\infty \psi'(t)(1 - F_n(t)) dt \leq C \int_\alpha^\infty \frac{\psi'(t)}{\varphi(t)} dt \leq C \int_\alpha^\infty \frac{\psi(t)}{\varphi(t)t} \frac{\varphi'(t)t}{\varphi(t)} dt.$$

We shall prove that $\varphi'(t)t/\varphi(t)$ is bounded so that the last integral is finite. Indeed, when φ' is increasing then (because of the Δ_2 -condition)

$$K\varphi(t) \geq \varphi(2t) = \int_0^{2t} \varphi'(\tau) d\tau > \int_t^{2t} \varphi'(\tau) d\tau > t\varphi'(t)$$

and in the case φ' is decreasing

$$\varphi(t) = \int_0^t \varphi'(\tau) d\tau > \int_{t/2}^t \varphi'(\tau) d\tau > \frac{1}{2}t\varphi'(t). \quad \text{Q.E.D.}$$

REMARKS. From the Proposition 2 and its proof it follows that

(a) if $\varphi, \psi \in \Delta_2$ have both monotone derivatives, $\psi(t)\varphi^{-1}(t) \rightarrow 0$ as $t \rightarrow +\infty$ in such a way that $\psi'(t)/\varphi(t) \in L^1(\alpha, \infty)$ for some $\alpha > 0$, and $\{X_n\}$ is upper weakly φ -finite then $\limsup_n E\psi(|X_n|) < \infty$.

(b) If $\{X_n\}$ is upper weakly φ -finite with $\varphi(u) = u^p, p > 0$, then

$$\limsup_n E|X_n|^q < \infty$$

for every $q, 0 < q < p$.

In what follows we shall need the following condition for the function φ :

$$(+) \quad 0 < \text{const} \leq \varphi(t)\varphi(t^{-1}) \leq \text{CONST} < \infty.$$

As examples of such a function we may give $\varphi(t) = t^p, p > 0$, and

$$\begin{aligned} \varphi(t) &= \int_0^t \frac{|\sin u|}{u} du && \text{for } 0 < t \leq \pi, \\ &= \varphi(\pi) + \int_\pi^t |\sin u| du && \text{for } \pi < t. \end{aligned}$$

THEOREM 1. *Let $\varphi \in \Delta_2$ satisfy condition (+). In this case if $\{X_n\}$ is l_φ -compatible then it is lower weakly φ -finite. If in addition φ' exists and is*

monotone then l_φ -compatibility of $\{X_n\}$ implies that $\varphi'(t) \liminf_n \int_{-t}^t s dF_n(s)$ is bounded for $0 \leq t < \infty$.

PROOF. Assume that $\{F_n\}$ is not lower weakly φ -finite. Then for every $k=1, 2, \dots$, we can find M_k such that

$$\min(\varphi(M_k), M_k) \geq k^2 \quad \text{and} \quad \varphi(M_k) \liminf_n P(|X_n| \geq M_k) > k.$$

It follows that for every $k=1, 2, \dots$, there exists an integer N_k such that

$$(*) \quad P(|X_j| \geq M_k) > k/\varphi(M_k), \quad j \geq N_k.$$

Let now $a_j=1/M_k$ for $m_k \leq j < m_k+r_k$, where $m_1=N_1$,

$$m_{k+1} = \max(m_k + r_k, N_{k+1})$$

and where r_k is an integer such that

$$(**) \quad 1 \leq k^2 r_k / \varphi(M_k) \leq 2,$$

and $a_j=0$ for j 's not listed above (if any). We shall check that $(a_j) \in l_\varphi$. Indeed

$$\sum_j \varphi(|a_j|) = \sum_k r_k \varphi\left(\frac{1}{M_k}\right) \leq \text{CONST} \sum_k \frac{r_k}{\varphi(M_k)} \leq \text{CONST} \sum_k \frac{2}{k^2} < \infty$$

because of the condition (+). However (*) and (**) cause the series

$$\sum_j P(|a_j X_j| \geq 1) = \sum_{k=1}^\infty \sum_{j=m_k}^{m_k+r_k-1} P(|X_j| \geq M_k) \geq \sum_{k=1}^\infty \frac{r_k k}{\varphi(M_k)} \geq \sum_{k=1}^\infty \frac{1}{k}$$

to be divergent, causing the divergence of $\sum_j a_j X_j$ in probability by the Kolmogorov three series theorem.

Now assume φ' is monotone and $\varphi'(t) \liminf_n \int_{-t}^t s dF_n(s)$ is unbounded on the positive half-line. Then, as in the preceding part of the proof, for every $k=1, 2, \dots$, we can find M_k, N_k such that $\min(\varphi(M_k), M_k) \geq k^2$ and

$$(***) \quad I_{jk} = \varphi'(M_k) \int_{|t| \leq M_k} s dF_j(s) > k, \quad j \geq N_k$$

(or $-I_{jk} > k$; in this case we proceed similarly). Having chosen (a_j) as previously in l_φ , we see in view of (***) that²

$$\begin{aligned} \sum_j E(a_j X_j)^1 &= \sum_{k=1}^\infty \frac{1}{M_k} \sum_{j=m_k}^{m_k+r_k-1} \int_{-M_k}^{M_k} s dF_j(s) > \sum_{k=1}^\infty \frac{r_k k}{M_k \varphi'(M_k)} \\ &\geq \frac{1}{\max(2, K)} \sum_{k=1}^\infty \frac{r_k k}{\varphi(M_k)} \geq \frac{1}{\max(2, K)} \sum_{k=1}^\infty \frac{1}{k}, \end{aligned}$$

² $X^1(\omega)$ is equal to $X(\omega)$ if $|X(\omega)| \leq 1$ and 0 otherwise.

the penultimate inequality being motivated as in the proof of the Proposition 2 by the Δ_2 -condition and the monotonicity of φ' . Thus again by the Kolmogorov three series theorem we conclude that $\{X_j\}$ is not l_φ -compatible. Q.E.D.

COROLLARY. *If $\sup_t \varphi'(t) = \infty$ (e.g. if $\varphi(t) = t^p, p > 1$) and $\{X_n\}$ is l_φ -compatible then $\liminf_n EX_n = 0$.*

THEOREM 2. *If $\{F_n\}$ is upper weakly φ -finite, where $\varphi(t) = t^p, 0 < p < 2$, and $t^{p-1} \sup_n \left| \int_{-t}^t s dF_n(s) \right| \leq C' < \infty, 0 \leq t < \infty$, then X_n is l_φ -compatible.*

PROOF. We know that $M^p \sup_n P(|X_n| > M) < C$ and we shall check the convergence of all three series in Kolmogorov's theorem. (a_i) always is supposed to belong to l_p .

$$(I) \quad \sum_i P(|a_i X_i| > 1) \leq \sum_i P\left(|X_i| > \frac{1}{|a_i|}\right) \leq \sum_i \sup_n P\left(|X_n| > \frac{1}{|a_i|}\right) \leq C \sum_i |a_i|^p < \infty.$$

$$(II) \quad \begin{aligned} \sum_i |E(a_i X_i)^1| &= \sum_i |a_i| \left| \int_{|t| < 1/|a_i|} s dF_i(s) \right| \\ &\leq \sum_i a_i \sup_n \left| \int_{|t| < 1/|a_i|} t dF_n \right| \\ &\leq C' \sum_i |a_i| |a_i|^{p-1} < \infty. \end{aligned}$$

$$(III) \quad \sum_i E((a_i X_i)^1)^2 = \sum_i a_i^2 \int_{|t| < 1/|a_i|} t^2 dF_i(dt).$$

It is sufficient to evaluate the series

$$\begin{aligned} \sum_i a_i^2 \int_0^{1/|a_i|} t^2 dF_i(t) &\leq \sum_i a_i^2 \left(\frac{1}{a_i^2} - 2 \int_0^{1/|a_i|} t F_i(t) dt \right) \\ &= 2 \sum_i a_i^2 \int_0^{1/|a_i|} t(1 - F_i(t)) dt \\ &\leq 2 \sum_i a_i^2 \left(1 + C \int_1^{1/|a_i|} t^{1-p} dt \right) \\ &= 2 \sum_i \left(1 - \frac{C}{2-p} \right) a_i^2 + \frac{2c}{2-p} \sum_i |a_i|^p < \infty. \end{aligned}$$

REMARKS. (1) If the random variables $\{X_n\}$ are identically distributed with distribution function F then the Theorems 1 and 2 give the following corollary which is the extension of R. Dudley's Theorem 7.2 of

[2]: if $0 < p < 2$ then X_n is l_p -compatible if and only if F is weakly t^p -finite (i.e. $M^p P(|X_1| \geq M)$ is bounded) and $t^{p-1} \int_{-t}^t s dF(s)$ is bounded.

(2) For $p > 1$ assumptions of Theorem 2 cause, clearly, $EX_n = 0$.

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DEPARTMENT OF MATHEMATICS, CARNEGIE-MELLON UNIVERSITY, PITTSBURGH, PENNSYLVANIA 15213

Current address: Institute of Mathematics, Wrocław University, Wrocław, pl. Grunwaldzki 2/4, Poland