PERMANENT GROUPS

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Abstract. A permanent group is a group of nonsingular matrices on which the permanent function is multiplicative. Let $A \circ B$ denote the Hadamard product of matrices $A$ and $B$. The set of groups $G$ of nonsingular $n \times n$ matrices which contain the diagonal group $D$ and such that for every pair $A$, $B$ of matrices in $G$ we have $A \circ B^T \in D$ is denoted by $\mathcal{A}_n$. If the underlying field has at least three elements then $\mathcal{A}_n$ consists of permanent groups. A partial converse is available: If a permanent group $G$ is generated by $D$ together with a set $S$ of elementary matrices and a set $Q$ of permutation matrices then $G = HK$ where $H$ is the subgroup generated by $Q$ and $K$ is generated by $D$ and $S$, and $K \in \mathcal{A}_n$.

1. Introduction. A group of nonsingular matrices on which the permanent function is multiplicative will be called a permanent group. In this paper we determine a large class of permanent groups.

It was conjectured by Marcus and Minc [2] that $\Delta_n$, the group of $n \times n$ nonsingular matrices of the form $PD$ is a maximal permanent group, where $P$ is any permutation matrix and $D$ is diagonal. The underlying field was unspecified in [2] and the conjecture was subsequently verified by the first author for the complex numbers [1]. Contrary to the assertion in [1], however, this is not the only maximal permanent group.

Let $A \circ B$ denote the Hadamard product of matrices $A$ and $B$. We shall denote by $\mathcal{A}_n(F)$ the collection of all groups $G$ of $n \times n$ nonsingular matrices over a field $F$ such that:

(a) $G$ contains the set of all $n \times n$ nonsingular diagonal matrices;
(b) if $A$ and $B$ are in $G$ then $A \circ B^T$ is a nonsingular diagonal matrix.

In Theorem 3.1 we will show that $\mathcal{A}_n(F)$ consists of permanent groups if $F$ has more than 2 elements. However, $\Delta_n$ is not in $\mathcal{A}_n(F)$.

The matrix $E_{ij}$ has $(i, j)$ entry 1 and zeros elsewhere. We will use elementary matrices $I + \lambda E_{ij}$, where $\lambda$ is a field element, $I$ the identity matrix, and $i \neq j$ for $i, j = 1, \ldots, n$. For any matrix $A$, $r_i(A)$ will denote...
the number of nonzero elements in row $i$. If row $i$ and column $i$ are deleted from a matrix $A$ the resulting matrix is denoted by $A(i|i)$.

2. Properties of $\mathcal{A}_n(F)$. We remark that $\mathcal{A}_n(F)$ contains both the permanent group of nonsingular upper triangular matrices and the permanent group of nonsingular lower triangular matrices. A less obvious permanent group is obtained by considering $n \times n$ upper triangular matrices $(a_{ij})$ over any field which satisfy the additional restrictions:

$$
\begin{align*}
a_{\alpha m} &= 0, & m &= \alpha + 1, \cdots, \beta, \\
a_{m\beta} &= 0, & m &= \alpha + 1, \cdots, \beta - 1,
\end{align*}
$$

for distinct integers $\alpha, \beta$ such that $1 \leq \alpha < \beta \leq n$.

If $P$ is the permutation matrix corresponding to the transposition $(\alpha \beta)$ and $H$ the group $\{I, P\}$ then the set $H \cdot K$ can be shown to be a permanent group, where $K$ consists of those nonsingular upper triangular matrices satisfying (2.1).

The following two lemmas will be used in proving that $\mathcal{A}_n(F)$ contains only permanent groups.

2.1 Lemma. If $A \in G \in \mathcal{A}_n(F)$, then $a_{ii} \neq 0$ for $i=1, \cdots, n$.

Proof. Suppose $A \in G \in \mathcal{A}_n(F)$, but $a_{ii}=0$ for some $i=1, \cdots, n$. Since $A$ is nonsingular there exists $y=1, \cdots, n$, such that $y_i$ and both $a_{iy}$ and its cofactor $A_{iy}$ are nonzero. Consequently, the matrix $A^{-1} \in G$ has a nonzero $(j, i)$ entry, contradicting $G \in \mathcal{A}_n(F)$.

2.2 Lemma. If $G \in \mathcal{A}_n(F)$ and $F$ has at least 3 elements then there exists a pair of positive integers $i, j$ $(1 \leq i, j \leq n)$ such that, for any $A \in G$,

$$
\begin{align*}
a_{im} &= 0, & m &\neq i, \\
a_{mj} &= 0, & m &\neq j.
\end{align*}
$$

Proof. The result is immediate when $n=2$ and the proof proceeds by induction on $n$.

Let $S$ be the set of all $i$ $(1 \leq i \leq n)$ for which there exists a matrix $A \in G$ such that $a_{ii} \neq 0$. Always, $1 \in S$. We shall show there is a $j$ $(1 \leq j \leq n)$ such that if $B$ is any matrix in $G$, then $b_{mj}=0$ for all $m \neq j$. If $S=\{1, \cdots, n\}$, then $G \in \mathcal{A}_n(F)$ implies $j$ may be chosen as 1. Otherwise, suppose $B$ is a matrix in $G$ such that $b_{ij} \neq 0$, and let $D=\text{diag}[1, \cdots, x, \cdots, 1]$ where $x$ is the $(i, i)$ entry. Consider an arbitrary matrix $C$ in $G$. Then $BDC$ is in $G$ and its $(1, j)$ entry is

$$
\sum_{k=1; k \neq i}^{n} b_{1k}c_{kj} + b_{1i}c_{ij}x.
$$
If $c_{ij}$ is nonzero, then (2.3) vanishes for at least one $x$ in $F$. Since $F$ has at least 3 elements we may always choose a nonzero $x$ in $F$ such that (2.3) is also nonzero. This implies that $c_{ij}=0$ whenever $i \in S$ and $j \in S'$, the complement of $S$ in $\{1, \ldots, n\}$.

Since $C$ was arbitrary in $G$ there is a permutation matrix $P$ such that, for any $A \in G$,

$$P^{-1}AP = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$$

where $A_1$ is $k \times k$ ($k$ is the cardinality of $S'$).

The set $\{P^{-1}AP | A \in G\}$ is also a group and hence $\{A_1 | A \in G\}=G_1$ is a group of $k \times k$ matrices. Evidently $G_1 \in S_n(F)$. Hence, by induction, each $A_1 \in G_1$ satisfies (2.2) for $m=1, \ldots, k$. Hence $P^{-1}AP$ satisfies

$$(P^{-1}AP)_{mj}=0, \quad m \neq j, m=1, \ldots, n.$$ 

Thus, there exists $j'$, not necessarily $j$, such that

$$a_{mj'}=0, \quad m \neq j', m=1, \ldots, n.$$ 

A similar argument yields an $i'$ such that

$$a_{im'}=0, \quad m \neq i', m=1, \ldots, n.$$ 

3. **The main theorem.**

3.1 **Theorem.** If $F$ is a field with at least 3 elements, then every group in each $S_n(F)$ is a permanent group.

This is an immediate consequence of the following two lemmas:

3.2 **Lemma.** If $A, B \in G \in S_n(F)$, then the $i$th diagonal entry of $AB$ is $a_{ii}b_{ii}$, for all $i=1, \ldots, n$.

**Proof.** The $(i, i)$ entry of $AB$ is $\sum_{k=1}^{n} a_{ik}b_{ki}$. If $k \neq i$ and $a_{ik} \neq 0$, then $b_{ki} = 0$ since $B \in G \in S_n(F)$. On the other hand, if $k \neq i$ and $b_{ki} \neq 0$, then $a_{ik} = 0$.

3.3 **Lemma.** If $A \in G \in S_n(F)$, then $\text{per } A = \prod_{i=1}^{n} a_{ii}$.

**Proof.** This is obvious for $n=1, 2$. Assume the result for all $k<n$. By Lemma 2.2 there is a column of $A$, say the $j$th, where the only nonzero entry is the diagonal one. It follows that the matrices $A(j|j)$ form a group in $S_{n-1}(F)$. This group will contain the nonsingular diagonal matrices because $G$ does. By induction, $\text{per } A(j|j) = \prod_{i \neq j}^{n} a_{ii}$ and so $\text{per } A = a_{jj} \text{ per } A(j|j) = \prod_{i=1}^{n} a_{ii}$.

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4. A partial converse. Clearly not every permanent group $G$ is in some $\mathcal{A}_n(F)$, e.g., $\Delta_n \notin \mathcal{A}_n(F)$. But if a permanent group is generated by the diagonal group together with a subset of the elementary matrices and a collection $\mathcal{Q}$ of permutation matrices, then $G=H \cdot K$ where $H$ is the subgroup generated by $\mathcal{Q}$ and $K \in \mathcal{A}_n(F)$ is generated by the diagonal group together with the given set of elementary matrices (Theorem 4.3).

4.1 Lemma. If $G$ is a permanent group generated by the diagonal group together with a set of elementary matrices, then for all $A \in G$, $a_{ij} \neq 0$ implies $E_{ij}(\lambda) \in G$ for every $\lambda \in F$.

Proof. Write $A$ as $E_1 \cdots E_mD$ where $D$ is diagonal and each $E_i$ is an elementary matrix in $G$. The only way $a_{ij}$ can be nonzero is if there is a sequence $i=i_0, i_1, \cdots, i_k=j$ where $E_{ir_{i+1}}(\lambda) \in G$ for $r=0, 1, \cdots, k-1$. However $E_{ij}$ is the product of the following elementary matrices

$$E_{i_0i_1}(\lambda) E_{ir_{i+1}(1)} E_{i_0i_1}(-\lambda) E_{ir_{i+1}(-1)}.$$ 

The proof is complete because if any group of nonsingular matrices containing the diagonal group also contains an elementary matrix $E_{pq}(\alpha)$ then it contains $E_{pq}(\beta)$ for every $\beta \in F$.

4.2 Lemma. Any permanent group generated by the diagonal group and a set of elementary matrices is in $\mathcal{A}_n(F)$.

Proof. Suppose $a_{ij} \neq 0$ for some $A \in G$, where $G$ is a permanent group satisfying the hypotheses. We must show $b_{ji} = 0$ for all $B \in G$. If $b_{ji} \neq 0$ for some $B \in G$ then the previous lemma implies both $E_{ij}(\lambda)$ and $E_{ji}(\lambda)$ are in $G$. However,

$$\text{per}[E_{ij}(1) \cdot E_{ji}(1)] = 3 \quad \text{while} \quad \text{per } E_{ij}(1) = \text{per } E_{ji}(1) = 1,$$

contradicting the assumption that $G$ was a permanent group.

4.3 Theorem. If $G$ is a permanent group generated by the diagonal group, a set of elementary matrices $S$, and a nonempty set of permutation matrices $\mathcal{Q}$ then $G=H \cdot K$ where $H$ is the subgroup generated by $\mathcal{Q}$ and $K \in \mathcal{A}_n(F)$ is generated by the diagonal group and $S$.

Proof. By virtue of Lemma 4.2, $K$ is in $\mathcal{A}_n(F)$, and we need only show that $HK=KH$. If $P$ is any permutation matrix in $H$ and $E \in K$, then $PE=EP$, where $F=PEP^{-1}$ is readily seen to be in $K$ also.
REFERENCES


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