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ISOMORPHISM TYPES OF INFINITE SYMMETRIC GRAPHS

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ABSTRACT. Professor Bjarni Jónsson asked about the cardinality of isomorphism types of infinite symmetric graphs of order m, for each infinite cardinal m. We show that there are 2^m pairwise nonisomorphic infinite symmetric graphs of order m, for each infinite cardinal m.

A *symmetric graph* is an ordered pair $\langle U, F\rangle$ where F is a symmetric relation over the set U. The cardinality of U is referred to as the *order* of the graph. Professor Bjarni Jónsson stated in [**2**, p. 31] that, as far as we know, the cardinality of the class of all pairwise nonisomorphic infinite symmetric graphs of order m, for each infinite cardinal m, is unknown. Since F is a subset of $U\times U$, it is trivial to see that 2^m is an upper bound. In this paper, we shall settle this cardinality question by proving the following theorem.

THEOREM. *The cardinality of the isomorphism types of infinite symmetric graphs of order m is 2^m for each infinite cardinal m.*

We base our proof on the result by Professors Comer and LeTourneau [**1**] that *there are 2^m pairwise nonisomorphic 1-unary root algebras of order m, for each infinite cardinal m.* With each 1-unary root algebra $A=\langle U, F\rangle$, we associate the symmetric graph $\bar{A}=\langle U, F\rangle$ where $F=f\cup f^{-1}$. To complete the proof of our theorem, it is therefore sufficient to prove the following lemma.

LEMMA. *Let $\bar{A}=\langle U, F\rangle$ and $\bar{B}=\langle U, G\rangle$ be two symmetric graphs associated with two 1-unary root algebras $A=\langle U,f\rangle$ and $B=\langle U, g\rangle$, respectively. If $\bar{A}$ and $\bar{B}$ are two isomorphic symmetric graphs, then A and B are two isomorphic 1-unary root algebras.*

Let a be the fixed point of $A=\langle U,f\rangle$. Note that a is the *only* fixed point, i.e., it is the only element of U satisfying $f(a)=a$. Let us write $x\to^f y$ or

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$y \xleftarrow{f} x$ if $f(x)=y$. (1) For each point $x \in U$, there is *one and only one* arrow leaving x. (2) The loop arrow $a \xrightarrow{f} a$ is the *only* arrow leaving the fixed point a. (3) For each $x\ (\neq a) \in U$ there is a least positive integer n, depending on x, satisfying $f^n(x)=a$. We shall call n the *height* of x. Then, we have connecting arrows from x to a as follows:

$$x \xrightarrow{f} f(x) \xrightarrow{f} \cdots \xrightarrow{f} f^{n-1}(x) \xrightarrow{f} f^n(x) = a$$

with $f^{n-1}(x) \neq f^n(x)$, i.e., $f^{n-1}(x) \neq a$.

PROOF. Assume that ϕ is a symmetric graph isomorphism from $\bar{A} = \langle U, F\rangle$ onto $\bar{B} = \langle U, G\rangle$. We shall show that ϕ is a 1-unary root algebra isomorphism from $A = \langle U, f\rangle$ onto $B = \langle U, g\rangle$. Since ϕ already is a bijection on U, it suffices to show that $g(\phi(x)) = \phi(f(x))$ for each $x \in U$, or equivalently,

$$(*) \qquad \phi(x) \xrightarrow{g} \phi(f(x)) \quad \text{for each } x \in U.$$

In other words, it is sufficient to show that ϕ is arrow preserving.

Since $(x, f(x)) \in f \subseteq f \cup f^{-1} = F$ and since ϕ is a symmetric graph isomorphism from $\bar{A} = \langle U, F\rangle$ to $\bar{B} = \langle U, G\rangle$, we have $\phi(x, f(x)) \in G$. But, then

$$(\phi(x), \phi(f(x)) = \phi(x, f(x)) \in G = g \cup g^{-1},$$

which means

$$(4) \qquad \phi(x) \ \underset{\xleftarrow[g]{}}{\overset{\xrightarrow{g}}{\text{or}}} \ \phi(f(x))$$

in an obvious sense. Note that to prove (*) is to prove that the top arrow of (4) holds.

If $x=a$, then (4) coincides with $\phi(a) \xrightarrow{g} \phi(a)$ in either case which shows, first, that (*) is true in case $x=a$ and, second, that

$$(5) \qquad \phi(a) \text{ is the fixed point, say, } b \text{ of } \bar{B} = \langle U, G\rangle.$$

If $x \neq a$, let $n\ (\geqq 1)$ be the height of x. By our observations (3), (4) and (5), we have

$$\phi(x) \ \underset{\xleftarrow[g_1]{}}{\overset{\xrightarrow{g^1}}{\text{or}}} \ \phi(f(x)) \ \underset{\xleftarrow[g_2]{}}{\overset{\xrightarrow{g^2}}{\text{or}}} \ \cdots \ \underset{\xleftarrow[g_{n-1}]{}}{\overset{\xrightarrow{g^{n-1}}}{\text{or}}} \ \phi(f^{n-1}(x)) \ \underset{\xleftarrow[g_n]{}}{\overset{\xrightarrow{g^n}}{\text{or}}} \ \phi(f^n(x)) = b$$

with $\phi(f^{n-1}(x)) \neq \phi(f^n(x)) = b$. (The upper and lower subscripts of g are attached only for the convenience of ensuing quotations.) By our early observation (2) we see that g^n holds while g_n does not. Since g^n already is an arrow leaving the element $\phi(f^{n-1}(x))$ in $\bar{B}$, there can be no other arrow leaving $\phi(f^{n-1}(x))$ by observation (1). Hence, g^{n-1} must hold, while g_{n-1} does not. Similarly applying (1) over and over, we shall have

$$\phi(x) \xrightarrow{g} \phi(f(x)) \xrightarrow{g} \cdots \xrightarrow{g} \phi(f^{n-1}(x)) \xrightarrow{g} \phi(f^n(x)) = b$$

the first arrow (from the left) of which surely proves (*). This completes a proof of our lemma and, consequently, our theorem.

Recall [1] that each of the aforementioned Comer-LeTourneau 1-unary root algebras has only the trivial automorphism group. From this, the following is immediate.

COROLLARY. *There are 2^m pairwise nonisomorphic infinite symmetric graphs of order m, for each infinite cardinal m, each with only the trivial automorphism group.*

REFERENCES

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