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ZEROS OF $\zeta'(s)$ IN THE CRITICAL STRIP

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ABSTRACT. It is shown that the abscissa of convergence for the Dirichlet series $(-1)^k(1-2^{1-s})^{k+1}\zeta^{(k)}(s)$ is zero, where $\zeta(s)$ is the Riemann zeta function. This implies the existence of infinitely many zeros of $\zeta'(s)$ in the critical strip.

Let $s=\sigma+it$. It is known that all nonreal zeros of $\zeta^{(k)}(s)$, the kth derivative of the Riemann zeta function, lie in a vertical strip surrounding the critical strip (Spira [1], [2]), and that $N_k(T)$, the number of zeros of $\zeta^{(k)}(s)$, $0<t\leqq T$, satisfies (Berndt [5])

$$N_k(T) \sim (T/2\pi)\log T. \tag{1}$$

Bohr and Landau [6] showed that if a Dirichlet series converges for $\sigma>0$, then $N(\alpha, T)$, the number of zeros for $\sigma>\alpha$ and $0\leqq t\leqq T$, is $O(T)$ for $\alpha>\frac{1}{2}$. This covers the case of Dirichlet L-series, and hence of $L^{(k)}(s,\chi)$. For $N(\sigma, T)$ for $\zeta(s)$, Bohr and Landau applied their theorem to $(1-2^{1-s})\zeta(s)$. In the present paper, we show how the Bohr-Landau theorem can be applied to $\zeta^{(k)}(s)$, by considering the function $(-1)^k(1-2^{1-s})^{k+1}\zeta^{(k)}(s)$ (a method which is in the folklore). A consequence of this result by (1) is that the critical strip contains infinitely many zeros of $\zeta'(s)$, and indeed that most of the zeros of $\zeta'(s)$ will lie in the strip $(0, \frac{1}{2}+\delta)$ for every $\delta>0$. On the Riemann hypothesis this will be true for every strip $[\frac{1}{2}, \frac{1}{2}+\delta)$ (Spira [4]).

THEOREM. *The abscissa of convergence of*

$$(-1)^k(1-2^{1-s})^{k+1}\zeta^{(k)}(s)$$

is zero for $k\geqq 0$.

PROOF. For $k=0$, the result is known. Integrating by parts, we have $\int \log^k u\, du=uP_k(\log u)$ where P_k is a monic polynomial of degree k. Next, an easy computation gives

$$\sum_{n\leqq x}\log^k n = \int_1^x \log^k u\, du + O(\log^k x), \qquad x \geqq 1.$$

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Let $(-1)^k(1-2^{1-s})^{k+1}\zeta^{(k)}(s)=\sum_{n=1}^{\infty}a_n n^{-s}$. Then

$$a_n=\sum_{\nu=0}^{\min(k+1,j)}\binom{k+1}{\nu}(-2)^\nu\left(\log\frac{n}{2^\nu}\right)^k \quad \text{if} \quad 2^j \,\|\, n$$

(j=the exact power of 2 dividing n). Thus,

$$\begin{aligned}\sum_{n=1}^{N}a_n&=\sum_{\nu=0}^{k+1}\binom{k+1}{\nu}(-2)^\nu\sum_{n\leq N;2^\nu|n}\left(\log\frac{n}{2^\nu}\right)^k\\&=\sum_{\nu=0}^{k+1}\binom{k+1}{\nu}(-2)^\nu\sum_{m\leq N/2^\nu}(\log m)^k\\&=\sum_{\nu=0}^{k+1}\binom{k+1}{\nu}(-2)^\nu\left\{\frac{N}{2^\nu}P_k\left(\log\frac{N}{2^\nu}\right)-P_k(0)+O(\log^k N)\right\}.\end{aligned}$$

Now, $\sum_{\nu=0}^{k+1}\binom{k+1}{\nu}(-1)^\nu P_k(\log x-\nu\log 2)$ is the $(k+1)$th difference of a polynomial in ν of degree k, and so is 0. Hence,

$$s_N=\sum_{n=1}^{N}a_n=O(\log^k N).$$

If $\sum a_n$ converges, then the abscissa of convergence is 0 (Titchmarsh [7, Chapter 9]) as the abscissa of absolute convergence is 1. If $\sum a_n$ diverges, then the abscissa of convergence is $\geqq 0$ and $\leqq \limsup_{N\to\infty}\log|s_N|/\log N\leqq \limsup_{N\to\infty}(\log k+\log\log N)/\log N=0$. So the abscissa of convergence is 0, proving the theorem.

From the results for $k=1$ of this paper, it appears highly unlikely that one could show a small strip $[\frac{1}{2},\frac{1}{2}+\delta)$ free of zeros of $\zeta'(s)$, and thus show the simplicity of the zeros by such a method.

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