NONNEGATIVE MATRICES WITH NONNEGATIVE INVERSES

RALPH DEMARR

Abstract. We generalize a result stating that a nonnegative finite square matrix has a nonnegative inverse if and only if it is the product of a permutation matrix by a diagonal matrix. We consider column-finite infinite matrices and give a simple proof using elementary ideas from the theory of partially ordered linear algebras.

In [1] the authors show that a nonnegative square matrix has a nonnegative inverse if and only if its entries are all zero except for a single positive entry in each row and column. In this note we generalize this result and simplify the proof as well.

Let $A$ denote the real linear algebra of all column-finite infinite matrices with real entries. We partially order $A$ as follows: $[a_{ij}] \leq [b_{ij}]$ if and only if $a_{ij} \leq b_{ij}$ for all $i, j$. Thus, $A$ is a partially ordered linear algebra (pola) and if $I$ denotes the identity matrix, then $0 \leq I$. See [2] for the precise definition of a pola. An example will illustrate the result to be obtained. Let $x = [a_{ij}]$ and $y = [b_{ij}]$ be defined as follows: $a_{ij} = 1$ if $i = j + 1$ and is zero otherwise; $b_{ij} = 1$ if $j = i + 1$ and is zero otherwise. Thus, $0 \leq x$, $0 \leq y$ and $0 \leq xy \leq 1 \leq yx$. Note that each column of $x$ contains exactly one positive entry and each row of $x$ contains at most one positive entry.

Theorem. Let $A$ be the pola described above. If $x, y \in A$, $0 \leq x$, $0 \leq y$ and $0 \leq xy \leq 1 \leq yx$, then each column of $x$ contains exactly one positive entry and each row of $x$ contains at most one positive entry. The conclusion applies to the matrix $y$ if we interchange the words “row” and “column”.

Proof. Define $d = yx - 1 \geq 0$ and note that $1 + d \leq (1 + d)^2 = yxyx \leq xy = 1 + d$ since $xy \leq 1$. Hence, $1 + 2d \leq (1 + d)^2 \leq 1 + d$, which means $d \leq 0$. Thus $d = 0$ and $yx = 1$, which means that $y$ is a left inverse for $x$. Hence, each column of $x$ must contain at least one positive entry. Next construct a matrix $z$ so that $0 \leq z \leq x$ and each column of $z$ has only one positive entry and this entry is equal to the corresponding entry in the matrix $x$. 

Received by the editors October 1, 1971.

Key words and phrases. Matrix theory, inverses, ordered algebras.

American Mathematical Society 1972
Note that $0 \leq zy \leq xy \leq 1$, which means that $zy$ and $xy$ are diagonal matrices. Hence, $(zy)(xy) = (xy)(zy)$. Now $z = (zy)(xy)x = (xy)(zy)x = x(yz)$ and $0 \leq yz \leq xy \leq 1$, which means that $yz$ is a diagonal matrix. Using elementary properties of matrix multiplication and the fact that $x$ and $z$ have one positive entry in common in each column we see that $yz = 1$ and therefore $x = z$. Hence, $x$ has exactly one positive entry in each column.

The example above shows that some rows of $x$ may contain only zeros. We show that $x$ has at most one positive entry in each row. Let us now construct a matrix $w$ so that $0 \leq w \leq x$ and each row of $w$ has only one positive entry if the same row of $x$ has a positive entry in it and this entry is equal to the corresponding entry in the matrix $x$. Now $w = (wy)x$ and since $0 \leq wy \leq xy \leq 1$, we see that $wy$ is a diagonal matrix. The same reasoning applied above to the matrix $z$ shows that $w = x$. Hence, $x$ has at most one positive entry in each row.

References


Department of Mathematics, University of New Mexico, Albuquerque, New Mexico 87106