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OPERATORS ASSOCIATED WITH A PAIR OF NONNEGATIVE MATRICES

GERALD E. SUCHAN

ABSTRACT. Let $A_{m\times n}$, $B_{m\times n}$, $X_{n\times 1}$, and $Y_{m\times 1}$ be matrices whose entries are nonnegative real numbers and suppose that no row of A and no column of B consists entirely of zeroes. Define the operators U, T and T' by $(UX)_i=X_i^{-1}$ [or $(UY)_i=Y_i^{-1}$], $T=UB^tUA$ and $T'=UAUB^t$. T is called irreducible if for no nonempty proper subset S of $\{1,\cdots,n\}$ it is true that $X_i=0, i\in S$; $X_i\neq 0$, $i\notin S$, implies $(TX)_i=0$, $i\in S$; $(TX)_i\neq 0$, $i\notin S$. M. V. Menon proved the following Theorem. If T is irreducible, there exist row-stochastic matrices A_1 and A_2, a positive number θ, and two diagonal matrices D and E with positive main diagonal entries such that $DAE=A_1$ and $\theta DBE=A_2^t$. Since an analogous theorem holds for T', it is natural to ask if it is possible that T' be irreducible if T is not. It is the intent of this paper to show that T' is irreducible if and only if T is irreducible.

Suppose that each of m and n is a positive integer. Let $A_{m\times n}$ and $B_{m\times n}$ be matrices whose entries are nonnegative real numbers and suppose that no row of A and no column of B consists entirely of zeroes. Let $X_{n\times 1}$ and $Y_{m\times 1}$ be matrices whose entries are taken from the extended real nonnegative numbers. Define the operator U by $(UX)_i=X_i^{-1}$ [or $(UY)_i=Y_i^{-1}$] and let $0^{-1}=\infty$, $\infty^{-1}=0$, $\infty+\infty=\infty$, $0\cdot\infty=0$, and if $\alpha>0$, $\alpha\cdot\infty=\infty$ [**1**]. Define the operators T and T' by $T=UB^tUA$ and $T'=UAUB^t$ where B^t is the transpose of B. Clearly

$$(TX)_i = \left(\sum_{j=1}^{m} b_{ji}\left(\sum_{k=1}^{n} a_{jk}X_k\right)^{-1}\right)^{-1}.$$

T is called irreducible if for no nonempty proper subset S of $N=\{1,\cdots,n\}$ is it true that $X_i=0$, $i\in S$; $X_i\neq 0$, $i\notin S$, implies $(TX)_i=0$, $i\in S$; $(TX)_i\neq 0$, $i\notin S$. T' is defined to be irreducible analogously.

M. V. Menon [**2**] proved the following.

THEOREM 1. *If T is irreducible, then there exist row-stochastic matrices A_1 and A_2, a positive number θ, and two diagonal matrices D and E with positive main diagonal entries such that $DAE=A_1$ and $\theta DBE=A_2^t$.*

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Since an analogous theorem holds for T', one would hope that T' might be irreducible even if T is not. However, it is the intent of this paper to prove the following result.

Theorem 2. *T' is irreducible if and only if T is irreducible.*

Proof. If T' is not irreducible then there is a nonempty proper subset S of $M=\{1,\cdots,m\}$ which has the property that if $Y_{m\times 1}$ is such that $Y_i=0$, $i\in S$, $Y_i\neq 0$, $i\notin S$, then $(T'Y)_i=0$, $i\in S$, $(T'Y)_i\neq 0$, $i\notin S$. Let $Z_{n\times 1}=UB^tY$, put $E=\{i\in N:Z_i=\infty\}$, and let E' be the complement of E in N.

(1) If $i_0\in S$ then there exists $j_0\in N$ such that $a_{i_0j_0}(\sum_{k=1}^m b_{kj_0}Y_k)^{-1}=\infty$. Hence $Z_{j_0}=\infty$ and thus E is not null.

(2) Let S' be the complement of S in M so that if $i_0\in S'$, then there exists $j_0\in N$ such that $\infty>a_{i_0j_0}(\sum_{k=1}^m b_{kj_0}Y_k)^{-1}>0$. Hence $\infty>Z_{j_0}>0$ and E' is not null.

(3) Let $X_{n\times 1}$ be defined by putting $\infty>X_i>0$ if $i\in E$ and $X_i=0$ if $i\in E'$. If $X_{i_0}=0$ then $\infty>(\sum_{j=1}^m b_{ji_0}(\sum_{k=1}^m a_{jk}Z_k)^{-1})^{-1}>0$ and hence there exists $j_0\in M$ such that $\infty>b_{j_0i_0}>0$. Thus $\infty>\sum_{k=1}^n a_{j_0k}Z_k=\sum_{k\in E}a_{j_0k}Z_k+\sum_{k\in E'}a_{j_0}kZ_k>0$ so that $a_{j_0k}=0$ for $k\in E$. Therefore, if $X_{i_0}=0$, then $(TX)_{i_0}=0$.

(4) For $i_0\in E$, put $F=\{j\in M:b_{ji_0}=0\}$ and let F' be the complement of F in M. Since $\infty>X_{i_0}>0$ then $(TZ)_{i_0}=\infty$ so that

$$\sum_{j=1}^m b_{ji_0}\left(\sum_{k=1}^n a_{jk}Z_k\right)^{-1}=\sum_{j\in F}b_{ji_0}\left(\sum_{k=1}^n a_{jk}Z_k\right)^{-1}+\sum_{j\in F'}b_{ji_0}\left(\sum_{k=1}^n a_{jk}Z_k\right)^{-1}=0,$$

and hence there exists $j_0\in F'$ so that $b_{j_0i_0}\neq 0$. Thus $\sum_{k=1}^n a_{j_0k}Z_k=\infty$ so that there exists $k_0\in E$ such that $\infty>a_{j_0k_0}X_{k_0}>0$. Therefore if $\infty>X_{i_0}>0$ then $\infty>(TX)_{i_0}>0$.

It immediately follows from (1), (2), (3), and (4) that T' is irreducible if T is irreducible. A similar argument proves that T is irreducible if T' is irreducible.

References

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Department of Mathematics, University of Houston, Houston, Texas 77004