THE RADIUS OF CLOSE-TO-CONVEXITY OF FUNCTIONS OF BOUNDED BOUNDARY ROTATION

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Abstract. An analytic function whose boundary rotation is bounded by $k\pi$ ($k \geq 2$) is shown to map a disc of radius $r_k$ onto a close-to-convex domain, where $r_k$ is the solution of a transcendental equation when $k > 4$ and $r_k = 1$ when $2 \leq k \leq 4$. The above value of $r_k$ is shown to be the best possible for each $k$ and an asymptotic expression for $r_k$ is obtained.

Let $V_k$ ($k \geq 2$) denote the class of functions $f(z)$ which are analytic in the unit disc $E = \{z : |z| < 1\}$, normalized by $f(0) = 0$ and $f'(0) = 1$, have non-vanishing derivatives in $E$, and map $E$ onto a domain which has boundary rotation at most $k\pi$. If $k = 2$, then $V_k$ is precisely the set of univalent functions which map $E$ onto a convex domain. If $2 < k \leq 4$, then $V_k$ is a subset of the functions which map $E$ onto a close-to-convex domain ([1], [6]). Finally, if $k > 4$, then functions in $V_k$ need not be close-to-convex or even univalent. In this paper we determine the radius of close-to-convexity of $V_k$ for each $k$. i.e. the radius of the largest disc centered at the origin which is mapped onto a close-to-convex domain by all $f$ in $V_k$. The techniques used are similar to those used by Krzyz in determining the radius of close-to-convexity of the class of univalent functions [2]. Some related problems were posed by M. O. Reade [5].

Theorem 1. If $k > 4$, the radius of close-to-convexity of $V_k$ is the unique root of the equation

$$(1) \quad 2 \cot^{-1}w - k \cot^{-1}(kw/2) = -\pi$$

in the interval $(R_k, 1)$ where $R_k$ is the radius of convexity of $V_k$ and $w = (1 - r^2)[k^2r^2 - (1 + r^2)^2]^{-1/2}$, while if $2 \leq k \leq 4$, the radius of close-to-convexity is 1.

Proof. Kaplan [1] has shown that a necessary and sufficient condition for a function $f(z)$, regular in $E$ and satisfying $f''(z) \neq 0$ to map $|z| = r$ onto
a close-to-convex curve is that

\[ \arg[z_2f'(z_2)] - \arg[z_1f'(z_1)] \geq -\pi \]

for all \( z_1 \) and \( z_2 \) with \( |z_1|=r \) and \( z_2=z_1e^{i\theta} \), \( 0<\theta<2\pi \). The radius of close-to-convexity of \( V_k \) is the largest value of \( r \) for which (2) holds for all \( f(z) \) in \( V_k \). The radius of convexity \( R_k \) of \( V_k \) is the smallest positive root of the equation \( 1-kr+r^2=0 \); \( R_2=1 \) and \( R_k<1 \) when \( k>2 \) [3]. Clearly the radius of close-to-convexity is larger than \( R_k \) when \( k>2 \) and equal to \( R_k \) when \( k=2 \), hence we assume throughout the remainder of this work that \( r>R_k \) and \( k>2 \).

Define

\[ \Delta(r, \theta) = \inf_{z \in V_k} \arg[z_2f'(z_2)/z_1f'(z_1)] \]

where \( z_1 \) and \( z_2 \) are defined as above and the argument is chosen to vary continuously from an initial value of zero. Let \( \zeta=(z-z_1)/(1-\bar{z}_1z) \) and \( \zeta_0=(z_2-z_1)/(1-\bar{z}_2z_2) \) and define \( g(\zeta) \) by

\[ g(\zeta) = [f((\zeta + z_1)/(1 + z_1\bar{z})) - f(z_1)]/f'(z_1)(1 - |z_1|^2). \]

Robertson has shown that \( g(z) \) is in \( V_k \) whenever \( f(z) \) is in \( V_k \) [7]. Evaluating \( g'(\zeta_0) \) directly yields

\[ g'(\zeta_0) = f'(z_2)(1 - \bar{z}_1z_2)^2/f'(z_1)(1 - |z_1|^2)^2; \]

hence we have \( \Delta(r, \theta) = \arg[(z_2/z_1)(1-\bar{z}_1z_2)^{-2}] + \inf_{g \in V_k} \arg[g'(\zeta_0)] \). Now

\[ \arg[(z_2/z_1)(1 - \bar{z}_1z_2)^{-2}] = 2 \cot^{-1}[(1 - r^2)\cot(\theta/2)/(1 + r^2)]; \]

\[ |\zeta_0| = r[2(1 - \cos \theta)/(1 - 2r^2 \cos \theta + r^4)]^{1/2}, \]

and

\[ \inf_{g \in V_k} \arg[g'(\zeta_0)] = -k \cot^{-1}[(1 - |\zeta_0|^2)^{1/2}/|\zeta_0|] \tag{4} \]

thus a brief calculation shows

\[ \Delta(r, \theta) = 2 \cot^{-1}[(1 - r^2)\cot(\theta/2)/(1 + r^2)] \]

\[ - k \cot^{-1}[(1 - r^2)/r(2(1 - \cos \theta))^{1/2}]. \tag{5} \]

Furthermore, this estimate is sharp since, for a fixed \( z_1 \) and \( z_2 \), if \( g(\zeta) \) is the function which gives equality in (4) and \( f(z) \) is defined by

\[ f(z) = [g((z - z_1)/(1 - \bar{z}_1z)) - g(-z_1)]/g'(-z_1)(1 - |z_1|^2), \]

then equality occurs in (3) for this choice of \( f(z) \). Let \( \Delta(b)=\inf \Delta(r, \theta) \) \((0<\theta<2\pi)\). Differentiating (5) with respect to \( \theta \) we obtain

\[ \partial \Delta(r, \theta)/\partial \theta = [1 + r^2 - kr \cos(\theta/2)](1 - r^2)/(1 - 2r^2 \cos \theta + r^4); \]
hence $\Delta(r, \theta)$ assumes its minimum value for a fixed $r$ when $\theta = \theta_0$ where $\cos(\theta_0/2) = (1+r^2)/kr$. The existence of $\theta_0$ is assured by the fact that for $r > R_k$, $(1+r^2)/kr < 1$. Substituting in (5), we have

$$\Delta(r) = 2 \cot^{-1}w - k \cot^{-1}(kw/2)$$

where $w = (1-r^2)(k^2r^2-(1+r^2)^2)^{-1/2}$. It is evident that $\Delta(r)$ is a decreasing function of $r$, hence $\Delta(r) \geq \Delta(1) = \pi(2-k)/2$. For $k \leq 4$, $\Delta(1) \geq -\pi$ and the radius of close-to-convexity is 1, while for $k > 4$, $\Delta(1) < -\pi$ and $\Delta(R_k) = 0$; hence there exists a unique solution $r_k$ to the equation $\Delta(r) = -\pi$, $R_k < r < 1$, and this solution is the radius of close-to-convexity.

Table 1 gives the approximate value of $r_k$ for various $k$. [The calculations were performed on a Univac 1106 by Mr. Michael Barnett of the Computer Science Center of Mankato State College.]

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**Theorem 2.** $\lim_{k \to \infty} kr_k = 2.9716 \ldots = \alpha$ where $\alpha$ is the unique root of the equation

$$\cot^{-1}[(\alpha^2 - 1)^{-1/2}] - (\alpha^2 - 1)^{1/2} = -\pi/2$$

in the interval $[\pi/2, \pi]$.

**Proof.** If $f(z)$ is in $V_k$, then (4) implies $\Re \{f'(z)\} > 0$ for $|z| < \pi/2k$. $\Re \{f'(z)\} > 0$ is a sufficient condition for close-to-convexity, hence $r_k \geq \pi/2k$. An examination of the mapping properties of the function

$$f_0(z) = (1/k)\{[(1 + z)/(1 - z)]^{k/2} - 1\}$$

shows that the radius of univalence $\rho_k$ of $f_0(z)$ satisfies $\rho_k = \csc(2\pi/k) - \cot(2\pi/k)$. Since $\lim k\rho_k = \pi$ ($k \to \infty$), we have $\alpha = \lim \sup kr_k \leq \pi$ ($k \to \infty$). If $\{k_n\}$ is any sequence such that $\lim k_n r_{k_n} = \alpha$ ($n \to \infty$), then it follows from (1) that $\alpha$ satisfies (7). However a differentiation of (7) shows the left-hand side to be a monotonic decreasing function and thus $\lim kr_n$ ($k \to \infty$) must exist and is the unique root of (7).
REFERENCES


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