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THE RADIUS OF CLOSE-TO-CONVEXITY OF FUNCTIONS OF BOUNDED BOUNDARY ROTATION

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ABSTRACT. An analytic function whose boundary rotation is bounded by $k\pi$ ($k \geqq 2$) is shown to map a disc of radius r_k onto a close-to-convex domain, where r_k is the solution of a transcendental equation when $k>4$ and $r_k=1$ when $2 \leqq k \leqq 4$. The above value of r_k is shown to be the best possible for each k and an asymptotic expression for r_k is obtained.

Let V_k ($k \geqq 2$) denote the class of functions $f(z)$ which are analytic in the unit disc $E=\{z:|z|<1\}$, normalized by $f(0)=0$ and $f'(0)=1$, have nonvanishing derivatives in E, and map E onto a domain which has boundary rotation at most $k\pi$. If $k=2$, then V_k is precisely the set of univalent functions which map E onto a convex domain. If $2<k\leqq 4$, then V_k is a subset of the functions which map E onto a close-to-convex domain ([**1**], [**6**]). Finally, if $k>4$, then functions in V_k need not be close-to-convex or even univalent. In this paper we determine the radius of close-to-convexity of V_k for each k, i.e. the radius of the largest disc centered at the origin which is mapped onto a close-to-convex domain by all f in V_k. The techniques used are similar to those used by Krzyż in determining the radius of close-to-convexity of the class of univalent functions [**2**]. Some related problems were posed by M. O. Reade [**5**].

THEOREM 1. *If $k>4$, the radius of close-to-convexity of V_k is the unique root of the equation*

$$2\cot^{-1}w - k\cot^{-1}(kw/2) = -\pi \tag{1}$$

in the interval $(R_k, 1)$ where R_k is the radius of convexity of V_k and $w=(1-r^2)[k^2r^2-(1+r^2)^2]^{-1/2}$, while if $2\leqq k\leqq 4$, the radius of close-to-convexity is 1.

PROOF. Kaplan [**1**] has shown that a necessary and sufficient condition for a function $f(z)$, regular in E and satisfying $f'(z)\neq 0$; to map $|z|=r$ onto

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a close-to-convex curve is that

$$\arg[z_2f'(z_2)] - \arg[z_1f'(z_1)] \geqq -\pi \tag{2}$$

for all z_1 and z_2 with $|z_1|=r$ and $z_2=z_1e^{i\theta}$, $0<\theta<2\pi$. The radius of close-to-convexity of V_k is the largest value of r for which (2) holds for all $f(z)$ in V_k. The radius of convexity R_k of V_k is the smallest positive root of the equation $1-kr+r^2=0$; $R_2=1$ and $R_k<1$ when $k>2$ [3]. Clearly the radius of close-to-convexity is larger than R_k when $k>2$ and equal to R_k when $k=2$, hence we assume throughout the remainder of this work that $r>R_k$ and $k>2$.

Define

$$\Delta(r, \theta) = \inf_{f\in V_k} \arg[z_2f'(z_2)/z_1f'(z_1)], \tag{3}$$

where z_1 and z_2 are defined as above and the argument is chosen to vary continuously from an initial value of zero. Let $\zeta=(z-z_1)/(1-\bar{z}_1z)$ and $\zeta_0=(z_2-z_1)/(1-\bar{z}_1z_2)$ and define $g(\zeta)$ by

$$g(\zeta) = [f(\{\zeta + z_1\}/\{1 + \bar{z}_1\zeta\}) - f(z_1)]/f'(z_1)(1 - |z_1|^2).$$

Robertson has shown that $g(z)$ is in V_k whenever $f(z)$ is in V_k [7]. Evaluating $g'(\zeta_0)$ directly yields

$$g'(\zeta_0) = f'(z_2)(1 - \bar{z}_1z_2)^2/f'(z_1)(1 - |z_1|^2)^2;$$

hence we have $\Delta(r, \theta)=\arg[(z_2/z_1)(1-\bar{z}_1z_2)^{-2}]+\inf \arg_{g\in V_k}[g'(\zeta_0)]$. Now

$$\arg[(z_2/z_1)(1 - \bar{z}_1z_2)^{-2}] = 2 \cot^{-1}[(1 - r^2)\cot(\theta/2)/(1 + r^2)],$$
$$|\zeta_0| = r[2(1 - \cos\theta)/(1 - 2r^2\cos\theta + r^4)]^{1/2},$$

and

$$\inf_{g\in V_k} \arg[g'(\zeta_0)] = -k \cot^{-1}[(1 - |\zeta_0|^2)^{1/2}/|\zeta_0|] \quad [4]; \tag{4}$$

thus a brief calculation shows

$$\begin{aligned}\Delta(r, \theta) = {} & 2 \cot^{-1}[(1 - r^2)\cot(\theta/2)/(1 + r^2)] \\ & - k \cot^{-1}[(1 - r^2)/r\{2(1 - \cos\theta)\}^{1/2}].\end{aligned} \tag{5}$$

Furthermore, this estimate is sharp since, for a fixed z_1 and z_2, if $g(\zeta)$ is the function which gives equality in (4) and $f(z)$ is defined by

$$f(z) = [g(\{z - z_1\}/\{1 - \bar{z}_1z\}) - g(-z_1)]/g'(-z_1)(1 - |z_1|^2),$$

then equality occurs in (3) for this choice of $f(z)$. Let $\Delta(r)=\inf \Delta(r, \theta)$ $(0<\theta<2\pi)$. Differentiating (5) with respect to θ we obtain

$$\partial\Delta(r, \theta)/\partial\theta = [1 + r^2 - kr\cos(\theta/2)](1 - r^2)/(1 - 2r^2\cos\theta + r^4);$$

hence $\Delta(r, \theta)$ assumes its minimum value for a fixed r when $\theta=\theta_0$ where $\cos(\theta_0/2)=(1+r^2)/kr$. The existence of θ_0 is assured by the fact that for $r>R_k$, $(1+r^2)/kr<1$. Substituting in (5), we have

$$\Delta(r) = 2\cot^{-1}w - k\cot^{-1}(kw/2) \tag{6}$$

where $w=(1-r^2)[k^2r^2-(1+r^2)^2]^{-1/2}$. It is evident that $\Delta(r)$ is a decreasing function of r, hence $\Delta(r)\geqq\Delta(1)=\pi(2-k)/2$. For $k\leqq 4$, $\Delta(1)\geqq-\pi$ and the radius of close-to-convexity is 1, while for $k>4$, $\Delta(1)<-\pi$ and $\Delta(R_k)=0$; hence there exists a unique solution r_k to the equation $\Delta(r)=-\pi$, $R_k<r<1$, and this solution is the radius of close-to-convexity.

Table 1 gives the approximate value of r_k for various k. [The calculations were performed on a Univac 1106 by Mr. Michael Barnett of the Computer Science Center of Mankato State College.]

TABLE 1

k	r_k	k	r_k	k	r_k
4	1	9	0.34593	50	0.05952
5	0.70388	10	0.30849	100	0.02973
6	0.55362	20	0.14994	200	0.01486
7	0.45961	30	0.09946	400	0.00743
8	0.39431	40	0.07446	800	0.00371

THEOREM 2. $\lim_{k\to\infty} kr_k=2.9716\ldots=\alpha$ *where* α *is the unique root of the equation*

$$\cot^{-1}[(\alpha^2-1)^{-1/2}] - (\alpha^2-1)^{1/2} = -\pi/2 \tag{7}$$

in the interval $[\pi/2, \pi]$.

PROOF. If $f(z)$ is in V_k, then (4) implies $\operatorname{Re}\{f'(z)\}>0$ for $|z|<\pi/2k$. $\operatorname{Re}\{f'(z)\}>0$ is a sufficient condition for close-to-convexity, hence $r_k\geqq\pi/2k$. An examination of the mapping properties of the function

$$f_0(z) = (1/k)\{[(1+z)/(1-z)]^{k/2} - 1\}$$

shows that the radius of univalence ρ_k of $f_0(z)$ satisfies $\rho_k=\csc(2\pi/k)-\cot(2\pi/k)$. Since $\lim k\rho_k=\pi$ $(k\to\infty)$, we have $\alpha=\limsup kr_k\leqq\pi$ $(k\to\infty)$. If $\{k_n\}$ is any sequence such that $\lim k_n r_{k_n}=\alpha$ $(n\to\infty)$, then it follows from (1) that α satisfies (7). However a differentiation of (7) shows the left-hand side to be a monotonic decreasing function and thus $\lim kr_k$ $(k\to\infty)$ must exist and is the unique root of (7).

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