PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, No. 1, September 1972

# PERTURBED ASYMPTOTICALLY STABLE SETS

ROGER C. McCANN

ABSTRACT. Perturbations of a dynamical system are defined and the behavior of compact asymptotically stable sets under these perturbations is determined. The occurrence of critical points in a perturbed planar dynamical system is also investigated.

In [1] it is shown that if $C$ is an asymptotically stable cycle of a planar dynamical system $\pi$ and if $\pi_i$ is a net of planar dynamical systems which converges to $\pi$, then there are limit cycles $C_i$ of $\pi_i$ such that $C_i \to C$. This paper presents a similar result in a more general setting for perturbed asymptotically stable sets. If $\pi_i$ is a net of dynamical systems which converges to a dynamical system $\pi$ and if $M$ is a compact asymptotically stable set of $\pi$, then eventually there are asymptotically stable sets $M_i$ of $\pi_i$ arbitrarily close to $M$. Moreover, if $M$ is invariant with respect to all $\pi_i$, then $M_i \to M$.

$R$, $R^+$, and $R^-$ will denote the reals, nonnegative reals, and nonpositive reals respectively.

A dynamical system $\pi$ on a topological space $X$ is a mapping of $X \times R$ onto $X$ which satisfies the following three conditions (where $x\pi t = \pi(x, t)$):

(i) $\pi$ is continuous in the product topology.

(ii) $x\pi 0 = x$ for each $x \in X$.

(iii) $(x\pi t)\pi s = x\pi(t+s)$ for each $x \in X$ and $s, t \in R$.

If $A \subset X$ and $B \subset R$, then $A\pi B$ will denote the set $\{x\pi t : x \in A, t \in B\}$. $L^+(x)$ and $L^-(x)$ will denote the positive limit set of $x$ and the negative limit set of $x$ respectively. A subset $M$ of $X$ is called an (negative) attractor iff there is a neighborhood $U$ of $M$ such that $(L^-(x))$ $L^+(x) \subset M$ for every $x \in U$. If $M$ is an (negative) attractor, then $(A^-(M))$ $A^+(M)$ will denote the largest such neighborhood.

A subset $S$ of $X$ is called a section with respect to $\pi$ iff $(S\pi t) \cap S = \varnothing$ for all $t \neq 0$.

In a topological space $X$ it is possible to define limits of nets of subsets

Received by the editors April 12, 1971 and, in revised form, January 7, 1972.

*AMS* 1970 *subject classifications*. Primary 34C35, 34D10; Secondary 34C05.

*Key words and phrases*. Dynamical systems, perturbation asymptotic stability, stable critical point.

$X_i \subset X$ as follows: let lim inf $X_i$ consist of all limits of nets of points $x_i \in X_i$; let lim sup $X_i$ consist of all limits of subnets of points $x_i \in X_i$. Obviously lim inf $X_i \subset$ lim sup $X_i$. If equality holds, the net $X_i$ is said to converge and we write

$$\lim X_i = \liminf X_i = \limsup X_i.$$

Let $\pi$ be a dynamical system on $X$ and $\pi_i$ be a net of dynamical systems on $X$ such that $\pi_i \to \pi$ in the following sense: if $x_j$ and $t_j$ are nets converging to $x$ and $t$ respectively, then $x_j\pi_i t_j \to x\pi t$ [**2**, VI, 3.1–3.11]. If $X$ is locally compact, then the convergence of $\pi_i$ to $\pi$ as defined above is equivalent to the convergence of $\pi_i$ to $\pi$ in the compact open topology [**2**, VI, 3.3]. A section with respect to $\pi$ may not be a section with respect to any of the $\pi_i$ [**2**, VI, 3.10.1].

*Convention.* Any set subscripted by an $i$ is to be considered relative to $\pi_i$; e.g., $L_i^+(x)$ is the positive limit set of $x$ with respect to $\pi_i$.

The purpose of this paper is to prove the following theorem.

THEOREM 1. *Let $X$ be a locally compact metric space on which there is defined a dynamical system $\pi$ and a net $\pi_i$ of dynamical systems such that $\pi_i \to \pi$. If $M$ is a compact asymptotically stable set of $\pi$, then there are asymptotically stable sets $M_i$ of $\pi_i$ such that* lim sup $M_i \subset M$. *Moreover, if $M$ is invariant with respect to each $\pi_i$, then* lim $M_i = M$.

Since $M$ is asymptotically stable there is a continuous Liapunov function $v: A^+(M) \to R^+$ for $M$ such that (i) $v(x\pi t) < v(x)$ whenever $x \notin M$ and $t > 0$ and (ii) $v(x) = 0$ whenever $x \in M$ [**3**, Theorem 10]. $\mathscr{F}$ will denote the family $\{v^{-1}([0, r]) : r > 0\}$, which is a fundamental system of neighborhoods of $M$. It is easily verified that $v^{-1}(r)$, $r > 0$, is a section with respect to $\pi$.

The proof of the theorem depends on the following two lemmas.

LEMMA 2. *Let $U, V \in \mathscr{F}$ be such that both are compact and $V \subset$ int $U$. Then eventually* $(\mathrm{cl}(X - U))\pi_i R^- \subset X - V$.

PROOF. Set $A = X - U$ and $B = X - V$. Evidently both are open and $\bar{A} \subset B$. Let $\alpha < 0$. By the construction of $U$, we have $\partial A\pi\alpha \subset A$ and $\bar{A}\pi_i[2\alpha, 0] \subset \bar{A} \subset B$. Since $\pi_i \to \pi$, eventually, say $i > i_0$, $\partial A\pi_i\alpha \subset A$ and $\bar{A}\pi[2\alpha, 0] \subset B$. We now show that $\bar{A}\pi_i R^- \subset B$ for $i > i_0$. Assume not. Then there is an $x \in \partial A$ and $t \in R^-$ such that $x\pi_i t \in \partial B$. Set $s = \inf\{\tau : x\pi_i\tau \in \partial A, t < s \leqq 0\}$. Then $t < s$ and $x\pi_i s \in \partial A$ since $\partial A$ is compact. Moreover, $x\pi_i(t, s) \cap \bar{A} = \varnothing$. Since $x\pi_i s \in \partial A$ and $x\pi_i t \in \partial B$, we have that $t - s < 2\alpha$ (recall $\bar{A}\pi_i[2\alpha, 0] \subset B$ for $i > i_0$). But $\partial A\pi_i\alpha \subset$ int $A$. This contradicts $\varnothing = x\pi_i(t, s) \cap \bar{A} = ((x\pi_i s)\pi(t - s, 0)) \cap \bar{A}$. This contradiction implies $\bar{A}\pi_i R^- \subset B$ for $i > i_0$.

LEMMA 3. *Let $U \in \mathcal{F}$ be compact. Then* $\mathrm{cl}(X-U)$ *is eventually a negative attractor with respect to* $\pi_i$.

PROOF. Let $V, W \in \mathcal{F}$ be compact and such that $V \subset \mathrm{int}\, U \subset U \subset \mathrm{int}\, W$ and set $A=X-W$, $B=X-U$, $C=X-V$. Each is open and $\bar{A} \subset B \subset \bar{B} \subset C$. Since $W$ and $V$ are in $\mathcal{F}$, for each $x \in \mathrm{cl}(C-A)$ there is a $t(x) \in R^-$ such that $x\pi t(x) \subset A$. We will first show that there is a $T \in R^-$ such that $x\pi[T, 0] \cap A \neq \varnothing$ for each $x \in \mathrm{cl}(C-A)$. Assume there is no such $T$. Then there are nets $x_i$ in $\mathrm{cl}(C-A)$ and $s(x_i)$ in $R^-$ such that $s(x_i) \to -\infty$ and $x_i\pi[s(x_i), 0] \cap A = \varnothing$. Since $\mathrm{cl}(C-A)$ is compact, we may assume that $x_i \to x \in \mathrm{cl}(C-A)$. Then $x\pi t(x) \in A$ and, since $A$ is open and $\pi$ continuous, $x_i\pi t(x) \in A$ eventually. This contradiction implies the existence of a $T \in R^-$ such that $x\pi[T, 0] \cap A \neq \varnothing$ for every $x \in \mathrm{cl}(C-A)$. Since $\mathrm{cl}(C-A)$ is compact and $\pi_i \to \pi$, eventually, say $i > i_0$, $x\pi_i[T, 0] \cap A \neq \varnothing$ for each $x \in \mathrm{cl}(C-A)$. $C$ is a neighborhood of $B$. $\bar{A}\pi_i R^- \subset B$ eventually (Lemma 2) so that $L_i^-(A) \subset B$. If $x \in \mathrm{cl}(C-A)$, then for $i > i_0$, $x\pi_i[T, 0] \cap A \neq \varnothing$ and (by Lemma 2) $L_i^-(\mathrm{cl}(C-A)) \subset B$ eventually. Thus eventually $L_i^-(C) \subset B$ and $B$ is eventually a negative weak attractor with respect to $\pi_i$.

PROOF OF THEOREM 1. Let the notation be as in Lemma 3. $B$ is a negative attractor and $A_i^-(B)-B \subset U$. Therefore $\partial A_i^-(B) \subset U$ and $\partial A_i^-(B)$ is compact. Hence $X-A_i^-(B)$ is asymptotically stable with respect to $\pi_i$ [**4**, Theorem 3.10]. Thus we have shown that, for each $r$, $M(r)=X-v^{-1}([0, r])$ is eventually a negative attractor, $X-A_i^-(M(r))$ is asymptotically stable and $X-A_i^-(M(r)) \subset v^{-1}([0, r])$. Set $r_i = \inf\{r : M(r)$ is a negative attractor with respect to $\pi_i\}$ and let $0 \leqq \varepsilon_i \leqq r_i$ be such that $M(r_i+\varepsilon_i)$ is a negative attractor with respect to $\pi_i$. Finally set $M_i = X-A_i^-(M(r_i+\varepsilon_i))$. $M_i \subset v^{-1}([0, r_i+\varepsilon_i])$ and is asymptotically stable. Lemma 3 implies $r_i \to 0$. Hence $M_i \subset v^{-1}([0, r_i+\varepsilon_i]) \to v^{-1}(0) = M$, so that $\limsup M_i \subset M$. If $M$ is invariant with respect to each $\pi_i$, then $L_i^+(M) \subset M$ so that $M \subset X-A_i^-(M(r_i+\varepsilon_i)) = M_i$. It easily follows that $M_i \to M$. This completes the proof.

REMARK. It should be noted that the converse of Theorem 1 is false. That is, if $M_i$ are compact asymptotically stable sets of $\pi_i$ and if $M_i$ converges to a compact set $M$, then it does not necessarily follow that $M$ is asymptotically stable with respect to $\pi$. Let $\pi$ be a planar dynamical system with the origin as a center-focus and $C_n$ $(n=1, 2, \cdots)$ a sequence of external limit cycles which converge to the origin. $\mathrm{cl}(\mathrm{int}\, C_n)$ is asymptotically stable and $\lim \mathrm{cl}(\mathrm{int}\, C_n)$ is the origin. Finally for each positive integer $n$, set $\pi_n = \pi$. $\mathrm{cl}(\mathrm{int}\, C_n)$ is asymptotically stable with respect to $\pi_n$, but the origin is not asymptotically stable with respect to $\pi$.

We now assume that $X$ is the plane $R^2$ and investigate the occurrence of critical points in the $\pi_i$. We will prove the following theorem.

THEOREM 4. *Let $x$ be a stable isolated critical point of a planar dynamical system $\pi$. Then there are critical points $x_i$ of $\pi_i$ such that $x_i \to x$.*

The proof depends on the following three lemmas.

LEMMA 5. *Each stable isolated critical point possesses arbitrarily small neighborhoods bounded by either a cycle or a section with respect to $\pi$ which is a simple closed curve.*

PROOF. The proof follows immediately from [**2**, VIII, 4.1] and [**2**, VIII, 4.3].

LEMMA 6. *Let $x \in X$ possess a fundamental system $\mathscr{F}$ of neighborhoods whose boundaries are simple closed curves which are sections with respect to $\pi$. If $W$ is any neighborhood of $x$, then there is a neighborhood $V \subset W$ of $x$ such that eventually $L_i^+(V) \subset W$ or $L_i^-(V) \subset W$.*

PROOF. Let $U \in \mathscr{F}$. For $\varepsilon > 0$, $\bar{U}\pi[\varepsilon, +\infty) \subset \text{int } U$ or $\bar{U}\pi(-\infty, -\varepsilon] \subset U$ [**2**, VII, 4.8]. Hence $\mathscr{F}$ contains a fundamental system $\mathscr{G}$ of compact neighborhoods of $x$ which consists entirely of positively invariant sets or of negatively invariant sets. For definiteness we will assume $\mathscr{G}$ consists of positively invariant sets. Let $V$, $U \in \mathscr{G}$ be such that $\bar{V} \subset \text{int } U \subset \bar{U} \subset \text{int } W$. In a manner similar to that used in the proof of Lemma 2, it can be shown that eventually $V\pi_i R^+ \subset U$. Thus $L_i^+(V) \subset \bar{U} \subset W$. If $\mathscr{G}$ consists of negatively invariant sets, the proof is analogous.

LEMMA 7. *Let $x \in X$ possess a fundamental system $\mathscr{F}$ of neighborhoods whose boundaries are cycles of $\pi$. If $W$ is any simply connected neighborhood of $x$, then there are sets $V_i \subset W$ such that eventually either $L_i^+(V_i) \subset W$ or $L_i^-(V_i) \subset W$.*

PROOF. Let $U \in \mathscr{F}$ be such that $\bar{U} \subset \text{int } W$. Since $\partial U$ contains no critical points with respect to $\pi$, eventually $\partial U$ contains no critical points with respect to $\pi_i$ [**2**, VI, 3.7]. Let $x_0 \in \partial U$, $T$ be the fundamental period of $x_0$ with respect to $\pi$, and $S_i$ be transversals (local sections which are arcs) with respect to $\pi_i$ and which generate neighborhoods of $x$ ([**2**, VI, 2.12] and [**2**, VII, 1.6]). Let $0 < \varepsilon < \frac{1}{4}T$. Eventually $x_0\pi_i(0, \frac{1}{2}T] \cap S_i = \varnothing$, $x_0\pi_i[\frac{1}{2}T, \frac{3}{2}T] \cap S_i\pi_i[-\varepsilon, \varepsilon] \neq \varnothing$, and $x\pi_i[0, \frac{3}{2}T] \subset \text{int } W$. Set $t_i = \inf\{\tau : x\pi\tau \in S_i, \tau > \frac{1}{2}T\}$. Let $C_i$ be the simple closed curve composed of $x\pi_i[0, t_i]$ and the subarc of $S_i$ connecting $x$ and $x\pi_i t_i$. Clearly this can eventually be done. Finally set $V_i = \text{int } C_i$. Then $\bar{V}_i \subset W$ since $W$ is simply connected. $V_i$ is either positively invariant or negatively invariant [**2**, VII, 4.8]. Hence $L_i^+(V_i) \subset \bar{V}_i \subset W$ or $L_i^-(V_i) \subset \bar{V}_i \subset W$. This completes the proof.

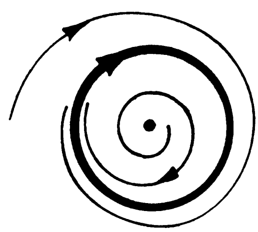
PROOF OF THEOREM 4. Let $x$ be a stable isolated critical point and $W$ be a compact simply connected neighborhood of $x$. By Lemmas 5, 6 and 7 there is a net $y_i$ in $W$ such that eventually either $L_i^+(y_i) \subset W$ or $L_i^-(y_i) \subset W$.

By the Poincaré-Bendixson Theorem [2, VIII, 1.14], if $L_i^+(y_i)\subset W$, then $L_i^+(y_i)$ is a cycle of $\pi_i$ or $L_i^+(y_i)$ contains critical points of $\pi_i$. A similar result holds if $L_i^-(y_i)\subset W$. If $L_i^+(y_i)$ (or $L_i^-(y_i)$) is a cycle then int $L_i^+(y_i)\subset W$ and int $L_i^+(y_i)$ contains a critical point [2, VII, 4.8]. Thus eventually $W$ contains critical points of $\pi_i$. Let $x_i$ be a critical point of $\pi_i$ and assume $x_i \to x_0$. Then, for any $t \in R$,

$$x_0 \leftarrow x_i = x_i\pi_i t \to x_0\pi t.$$

Hence $x_0$ is a critical point of $\pi$. Since $x$ is an isolated critical point of $\pi$, the desired result easily follows.

REMARKS. (1) If $x$ is a stable critical point of $\pi$ and $x_i$ are critical points of $\pi_i$ such that $x_i \to x$, it may be that none of the $x_i$ are stable. Let $\pi_n$ $(n=1, 2, \cdots)$ be the planar dynamical system indicated by the following drawing (where the cycle is a circle of radius $1/n$). Then the $\pi_n$ can be



chosen so that they converge to a dynamical system $\pi_n$ in which the origin is asymptotically stable.

(2) If $x$ is a critical point of $\pi$, it is possible that there are no critical points of the $\pi_i$ close to $x$. Let $\pi_n$ $(n=1, 2, \cdots)$ be the planar dynamical system given by $\dot{x}=x^2/(1+x^2)+1/n$, $\dot{y}=0$ and $\pi$ be the planar system given by $\dot{x}=x^2/(1+x^2)$, $\dot{y}=0$. Then $\pi_n \to \pi$, each $\pi_n$ is free from critical points, and $\pi$ has a critical point at the origin.

## REFERENCES


1. R. C. McCann, *Local sections of perturbed local dynamical systems*, J. Differential Equations **10** (1972), 336–344.

2. O. Hájek, *Dynamical systems in the plane*, Academic Press, New York, 1968. MR **39** #1767.

3. J. Auslander and P. Seibert, *Prolongations and stability in dynamical systems*, Ann. Inst. Fourier (Grenoble) **14** (1964), fasc. 2, 237–267. MR **31** #455.

4. N. P. Bhatia and O. Hájek, *Theory of dynamical systems*. II, Technical Note No. BN-606, University of Maryland, College Park, Md., 1969.


DEPARTMENT OF MATHEMATICS AND STATISTICS, CASE WESTERN RESERVE UNIVERSITY, CLEVELAND, OHIO 44106