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A NOTE ON THE COMPACT ELEMENTS OF C^*-ALGEBRAS

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ABSTRACT. It is shown that for any C^*-algebra A there is a faithful representation π of A on a Hilbert space H such that, for $u \in A$, the map $x \mapsto uxu$ is a compact operator on A if and only if $\pi(u)$ is a compact operator on H.

The purpose of this note is to point out how some recent results of J. A. Erdos [2] may be used to augment the author's study [5] of the compact elements of C^*-algebras.

An element u of a C^*-algebra A is called *compact*, if the mapping $x \mapsto uxu$ is a compact operator on A.

THEOREM. *Let A be a C^*-algebra. There exists an isometric $*$-representation π of A on a Hilbert space H such that $u \in A$ is a compact element of A if and only if $\pi(u)$ is a compact operator on H. Furthermore, the linear operator $x \mapsto uxu$ on A has finite rank if and only if $\pi(u)$ has finite rank.*

PROOF. We denote by C the set of the compact elements of A and set $F = \{u \in A \mid \text{the operator } x \mapsto uxu \text{ on } A \text{ has finite rank}\}$. Let us first show that there is an isometric $*$-representation π of A such that $\pi(u)$ is a compact operator for each $u \in C$, and $\pi(u)$ has finite rank if $u \in F$. If zero is the only compact element of A, this is obvious. If A contains a nonzero compact element, it follows from Theorems 3.10 and 5.1 in [5] that the socle of A in the sense of [3, p. 46] exists and coincides with F (see also [1, Theorem 7.2]), and its norm closure equals C. Therefore, by virtue of Theorem 3.7 and Lemma 4.1 in [2], there exists an isometric $*$-representation π of A on a Hilbert space H such that $\pi(u)$ has finite rank for each $u \in F$, and consequently $\pi(u)$ is a compact operator on H if $u \in C$. Conversely, if $\pi(u)$ is a compact operator (resp. has finite rank), it is a compact element of $\pi(A)$ (resp. the operator $T \mapsto \pi(u)T\pi(u)$ on $\pi(A)$ has finite rank) (see [4, Theorem 3] or [5, Theorem 7.5]), and it follows that $u \in C$ (resp. $u \in F$).

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