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THE NUMERICAL RANGE OF A TOEPLITZ OPERATOR[1]

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ABSTRACT. In this paper we explicitly compute the numerical range of an arbitrary Toeplitz operator on the classical Hardy space H^2 of the unit circle. In particular, we show that the numerical range depends only on the spectrum of the given Toeplitz operator. Several special cases are also considered.

If A is a bounded linear operator on a Hilbert space $(H, (\, , \,))$, the numerical range of A is the set $W(A)=\{(Af, f): \|f\|=1\}$. $W(A)$ is a convex set whose closure contains the spectrum, $\mathrm{sp}(A)$, of A. If A is normal, then A is convexoid, i.e., the closure of $W(A)$ is the convex hull of $\mathrm{sp}(A)$, and furthermore every extreme point of $W(A)$ is an eigenvalue of A. Proofs of these facts may be found in [**2**, Chapter 17] and [**4**].

Let m denote normalized linear Lebesgue measure on the unit circle in the complex plane. For $1 \leqq p \leqq \infty$ let H^p be the usual Hardy space of all functions f in $L^p=L^p(m)$ whose Fourier transform vanishes on the negative integers. Each f in H^p may be extended to a function (also denoted by f) holomorphic on the open unit disk. Let P be the orthogonal projection of L^2 onto H^2. If f is in L^∞, the Toeplitz operator T_f induced by f is defined by $T_f g=Pfg$, for g in H^2. Elementary properties of Toeplitz operators are discussed in [**1**] and will be used without further mention. In particular, Toeplitz operators are convexoid. In this paper we provide a complete description of the numerical range of an arbitrary Toeplitz operator. Some of the ideas involved were adumbrated in [**1**] but were not exploited to their full extent.

We first consider the case when T_f is normal.

THEOREM 1. *Let f be a nonconstant function in L^∞ and suppose that T_f is normal. Then $\mathrm{sp}(T_f)$ is a closed line segment $[a, b]$ and $W(T_f)$ is the corresponding open segment (a, b).*

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PROOF. A normal Toeplitz operator is a linear function of a selfadjoint Toeplitz operator, so there is a real function g in L^∞ and constants c, d such that $T_f = cT_g + d$. By a theorem of Hartman-Wintner [**2**, Problem 199], $\operatorname{sp}(T_g) = [m, M]$, where m is the essential infimum and M is the essential supremum of g. Hence $\operatorname{sp}(T_f) = [cm+d, cM+d]$. Now T_g is selfadjoint, hence convexoid, so $W(T_g)$ is a convex set whose closure is in $[m, M]$. Therefore, (m, M) is a subset of $W(T_g)$. If either m or M is in $W(T_g)$, it is an extreme point of $W(T_g)$, hence is an eigenvalue of T_g. A selfadjoint Toeplitz operator with an eigenvalue is constant [**2**, Problem 198]. Hence $f = cg + d$ is constant, which contradicts the hypothesis. It follows that $W(T_g) = (m, M)$ and $W(T_f) = (cm+d, cM+d)$.

COROLLARY 1. *If f is a nonconstant real function in L^∞, then* $W(T_f) =$ (ess inf(f), ess sup(f)).

When T_f is nonnormal, the situation is somewhat more complicated.

THEOREM 2. *Let f be in L^∞ and suppose that T_f is nonnormal. Then $W(T_f)$ is the interior of the convex hull of the spectrum of T_f.*

PROOF. Suppose that $W(T_f)$ is not open in the complex plane. Let z be a point in the boundary of $W(T_f)$ and in $W(T_f)$. Then 0 is in the boundary of $W(T_{f-z})$ and in $W(T_{f-z})$. Since $W(T_{f-z})$ is convex, we may rotate $W(T_{f-z})$ so that it lies in the right half-plane; i.e., there is a complex number b of unit modulus such that $W(T_{bf-bz})$ lies in the right half-plane and, of course, we still have the condition that 0 is in $W(T_{bf-bz})$. Therefore T_{bf-bz} has nonnegative real part and consequently $\operatorname{Re}(bf-bz)$ is nonnegative. By a result of Brown and Halmos [**1**, Theorem 10], $bf-bz = ig$, where g is real in L^∞. Hence $f = cg+d$ and $T_f = cT_g + d$ is normal, which contradicts the hypothesis. Therefore $W(T_f)$ is an open convex set whose closure is the convex hull of the spectrum of T_f. An open convex set is the interior of its closure, and the proof is complete.

Recall that if f is in H^∞, then the spectrum of T_f is the closure of the range of f on the open unit disk U.

COROLLARY 2. *If f is in H^∞, then the numerical range of T_f is the convex hull of $f(U)$.*

PROOF. We may assume that f is nonconstant. Then T_f is not normal (since $T_f^* = T_{\bar{f}}$ and two Toeplitz operators commute if and only if they are both analytic or both coanalytic or one is a linear function of the other), so $W(T_f)$ is the interior of the convex hull of the closure of $f(U)$. The convex hull of a compact set is compact, the convex hull of an open set is open, and an open convex set is the interior of its closure. From these facts it follows readily that $W(T_f)$ is the convex hull of $f(U)$.

COROLLARY 3. *If f in L^∞ is nonconstant, then the numerical range of T_f has no extreme points.*

It is not true in general that the spectrum of an operator determines the numerical range, not even if the operator is selfadjoint (and hence convexoid). That is, two selfadjoint operators can have the same spectrum but different numerical ranges. For example, if f is a nontrivial characteristic function in L^∞, then the spectrum of T_f is the closed unit interval [0, 1] and $W(T_f)$ is the open interval (0, 1) by Corollary 1. On the other hand, it is easy to construct a selfadjoint operator A on a separable Hilbert space such that $\mathrm{sp}(A)=W(A)=[0,1]$; simply let A be a diagonal operator with the rationals in [0, 1] for the diagonal. Hence if B is a convexoid operator we cannot, from $\mathrm{sp}(B)$, ascertain $W(B)$, but merely the closure of $W(B)$, which is the convex hull of $\mathrm{sp}(B)$.

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