

THE FOURIER TRANSFORM OF AN UNBOUNDED SPECTRAL DISTRIBUTION

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ABSTRACT. The Fourier transform of an unbounded spectral distribution is studied: An explicit integral representation is obtained; connections are drawn to the associated generalized scalar operator. It is proved that every generalized pseudo-hermitian operator is the infinitesimal generator of a temperate C_0 group.

Introduction. In [6] the author introduced a theory of unbounded spectral distributions in Banach spaces, and a corresponding theory of the generalized scalar operators which they represent. Properties of these objects were studied, culminating in a spectral mapping theorem [6, Theorem 4]. In this paper we study the Fourier transform of an unbounded spectral distribution, deriving an explicit integral representation, as well as growth estimates at infinity. The case of real support is considered in some detail, leading to the proof that every generalized pseudo-hermitian operator (i.e., generalized scalar with real spectrum) is the infinitesimal generator of a temperate group. This result generalizes the corresponding result for bounded operators proved in [5].

In this paper, all definitions are as in [6].

Integral representation of the Fourier transform.

THEOREM 1 (EXTENSION OF SPECTRAL DISTRIBUTIONS). *Let T be a spectral distribution [6, Definition 2] in a Banach space X . Then*

(a) *for each function f from the space $\mathcal{B}(R^2)$ of C^∞ functions with bounded derivatives, and for each $x \in X$, $\lim_n T_{f\varphi_n}x$ exists, where $\{\varphi_n\}$ is a sequence of test functions as in [6, Definition 2(c)].*

(b) *If we define $T_f x \equiv \lim_n T_{f\varphi_n}x$ for all $x \in X$, then T_f is in the space $\mathcal{L}(X)$ of all bounded linear operators on X , and the correspondence $f \rightarrow T_f$ is a continuous linear mapping of $\mathcal{B}(R^2)$ into $\mathcal{L}(X)$.*

(c) *$T_{fg} = T_f T_g$ for all $f, g \in \mathcal{B}(R^2)$.*

(d) *The operator T_f is independent of the sequence $\{\varphi_n\}$*

PROOF. (a) Define a double sequence of functions $\psi_{nm} \equiv f(\varphi_n - \varphi_m)$. Then $\{\psi_{nm}\}$ form a $\mathcal{B}(R^2)$ -bounded subset of $\mathcal{L}(R^2)$ (cf. [3, p. 91] for the

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topology of $\mathcal{B}(R^2)$). And $\inf\{|\lambda|: \lambda \in \text{support } \psi_{nm}\} \rightarrow \infty$ as $n, m \rightarrow \infty$. Since T strongly vanishes to power 0 at infinity (cf. [6, Definition 1]), $T_{f\varphi_n}x - T_{f\varphi_m}x = T_{\psi_{nm}}x \rightarrow 0$.

(b) T_f is everywhere defined and linear on X , by (a). Since $\|T_{f\varphi_n}\|$ are bounded, T_f is bounded. It is clear that the correspondence $f \rightarrow T_f$ is linear. Let \mathcal{F} be a bounded subset of $\mathcal{B}(R^2)$. Then the set $\{\varphi_n f: f \in \mathcal{F}\}$ is a $\mathcal{B}(R^2)$ -bounded subset of $\mathcal{D}(R^2)$. Thus $\|T_{f\varphi_n}\|$ are bounded; i.e., there is an $M > 0$ such that $\|T_{f\varphi_n}x\| \leq M\|x\|$ for all $x \in X, f \in \mathcal{F}$, and n . Letting $n \rightarrow \infty$, we get $\|T_f\| \leq M$ for all $f \in \mathcal{F}$. Since $\mathcal{B}(R^2)$ is bornological [3, p. 222], the correspondence $f \rightarrow T_f$ is continuous.

(c) $T_{f\varphi_n}x = \lim_m T_{f\varphi_n}T_{\varphi_m}x = \lim_m T_{f\varphi_n\varphi_m}x = \lim_m T_{f\varphi_n}T_{\varphi_m}x = T_{f\varphi_n}T_gx$. Taking the limit as $n \rightarrow \infty$, we get $T_{f\varphi_n}x = T_fT_gx$ for all $x \in X$.

(d) If $\{\psi_n\}$ is another such sequence, then $T_{f\varphi_n}x - T_{f\psi_n}x = T_{f(\varphi_n - \psi_n)}x$. Since $\{f(\varphi_n - \psi_n)\}$ is a $\mathcal{B}(R^2)$ -bounded subset of $\mathcal{D}(R^2)$ for which the supports tend to infinity, the conclusion follows from the fact that T strongly vanishes to power 0 at infinity.

REMARKS. For the function $f \in \mathcal{B}(R^2)$ which is identically 1, $T_f = I$, the identity operator in X , by the normalization condition on T [6, Definition 2(c)].

The extension of T to $\mathcal{B}_0(R^2)$ (functions in $\mathcal{B}(R^2)$ vanishing at infinity together with all derivatives) is unique as a continuous extension of T ; and to $\mathcal{B}(R^2)$, unique as a multiplicative extension.

Since T is continuous with respect to the $\mathcal{B}(R^2)$ topology, it is tempered. Thus T has a Fourier transform \hat{T} defined by the formula $\hat{T}_\varphi \equiv T_\varphi^\wedge$ for all $\varphi \in \mathcal{L}(R^2) \subset \mathcal{B}(R^2)$.

THEOREM 2. Let T be a spectral distribution in X . For $(\xi, \eta) \in R^2$, define

(i) $f_{\xi, \eta}(\lambda) \equiv \exp(-i(\text{Re } \lambda \cdot \xi + \text{Im } \lambda \cdot \eta))$ for all $\lambda \in R^2 \equiv C$.

Then

(a) $f_{\xi, \eta}(\cdot) \in \mathcal{B}(R^2)$;

(b) the function F from R^2 into $\mathcal{L}(X)$ defined by $F(\xi, \eta) \equiv T_{f_{\xi, \eta}}$ is strongly continuous in $(\xi, \eta) \in R^2$;

(c) $F(0, 0) = I$;

(d) $\hat{T}_\varphi x = \int_{R^2} \varphi(\xi, \eta) F(\xi, \eta) x \, d\xi \, d\eta$ for all $\varphi \in \mathcal{L}(R^2)$, $x \in X$;

(e) there is a $C > 0$ and a positive integer N such that

$$\|F(\xi, \eta)\| \leq C(1 + |\xi + i\eta|)^N \text{ for all } (\xi, \eta) \in R^2.$$

PROOF. (a) and (c) are clear.

(b) Since $F(\xi, \eta)x = \lim_n T_{\varphi_n f_{\xi, \eta}}x$, F is strongly Bochner measurable; in fact, the functions $F_n(\xi, \eta) \equiv T_{\varphi_n f_{\xi, \eta}}$ are even analytic. Let $G(\xi) \equiv F(\xi, 0)$. Then $G(\xi_1 + \xi_2) = G(\xi_1)G(\xi_2)$, and G is strongly measurable, so

Lemma 3 of [1, p. 616] implies G is strongly continuous. Similarly, $H(\eta) \equiv F(0, \eta)$ is strongly continuous. Now $F(\xi, \eta) = G(\xi)H(\eta) = H(\eta)G(\xi)$; let $x \in X$, $\xi_n \rightarrow \xi$, $\eta_n \rightarrow \eta$. Then $H(\eta_n)x$ are bounded, so the Banach-Steinhaus Theorem implies $\|H(\eta_n)\|$ are bounded by some number M . Thus

$$\begin{aligned} & \|F(\xi_n, \eta_n)x - F(\xi, \eta)x\| \\ & \leq \|H(\eta_n)[G(\xi_n) - G(\xi)]x\| + \|[H(\eta_n) - H(\eta)]G(\xi)x\| \\ & \leq M \| [G(\xi_n) - G(\xi)]x\| + \|[H(\eta_n) - H(\eta)]G(\xi)x\|, \end{aligned}$$

which implies the strong continuity of F .

(d) Fix $x \in X$, $x^* \in X^*$ (dual of X). Let $\{u_n\}$ be the distributions defined by $u_n(\varphi) = x^*T_{\varphi_n}x$ for all $\varphi \in \mathcal{D}(R^2)$, where φ_n are as usual. Then u_n has compact support, and [2, Theorem 1.7.5] implies

$$\begin{aligned} \hat{u}_n(\varphi) &= \int_{R^2} \varphi(\xi, \eta) u_n(f_{\xi, \eta}) d\xi d\eta \\ &= \int_{R^2} \varphi(\xi, \eta) [x^*F_n(\xi, \eta)x] d\xi d\eta \\ &\xrightarrow{n} \int_{R^2} \varphi(\xi, \eta) x^*F(\xi, \eta)x d\xi d\eta \end{aligned}$$

by the Dominated Convergence Theorem, since $\|F_n(\xi, \eta)x\| \leq C\|F(\xi, \eta)x\|$ (where $C = \sup_n \|T_{\varphi_n}\|$). Of course $\|F(\xi, \eta)x\|$ is bounded on the support of φ by (b). On the other hand $\hat{u}_n(\varphi) = u_n(\hat{\varphi}) = x^*T_{\varphi_n \hat{\varphi}}x \rightarrow_n x^*T_{\hat{\varphi}}x$. Thus

$$x^*T_{\hat{\varphi}}x = \int_{R^2} \varphi(\xi, \eta) x^*F(\xi, \eta)x d\xi d\eta = x^* \int_{R^2} \varphi(\xi, \eta) F(\xi, \eta)x d\xi d\eta$$

from which (d) is an immediate consequence.

(e) By the continuity of T on $\mathcal{B}(R^2)$ there is a constant C and a positive integer p such that

$$(ii) \quad \|T_{\varphi}\| \leq C \cdot \sum_{|\alpha| \leq p} |D^\alpha \varphi|_\infty \quad \text{for all } \varphi \in \mathcal{B}(R^2),$$

where $\alpha \equiv (\alpha_1, \alpha_2)$ is a bi-index, $|\alpha| \equiv \alpha_1 + \alpha_2$, and $D^\alpha \equiv \partial^{|\alpha|} / (\partial x_1)^{\alpha_1} (\partial x_2)^{\alpha_2}$. (Here $x_1 \equiv \text{Re } \lambda$, $x_2 \equiv \text{Im } \lambda$.)

Now

$$\begin{aligned} \|F(\xi, 0)\| &= \|T_{f_{\xi, 0}}\| \leq C \sum_{|\alpha| \leq p} |D^\alpha f_{\xi, 0}|_\infty \\ &= C \sum_{x_1=0}^p |\partial^{x_1} / \partial x_1^{x_1} \exp[-i\xi x_1]|_\infty = CP(|\xi|), \end{aligned}$$

where $P(t) \equiv \sum_{x_1=0}^p t^{x_1}$. Similarly $\|F(0, \eta)\| \leq CP(|\eta|)$. Thus $\|F(\xi, \eta)\| \leq C^2 P^2(|\xi + i\eta|)$, from which (e) follows.

REMARK. As a corollary of (e), we can replace the space $\mathcal{D}(R^2)$ by the space $\mathcal{S}(R^2)$ in part (d).

The next theorem shows that the analysis can be carried to complex values of η if the support of T is real, or equivalently, if T represents an operator with real spectrum (cf. [6, Theorem 4]).

THEOREM 3. *Let T be a spectral distribution in X which has real support. Then*

(a) *For $(\xi, \eta) \in R \times C$ (i.e., ξ real, η complex) there is a function $f_{\xi, \eta}(\cdot) \in \mathcal{B}(R^2)$ satisfying equation (i) near $\text{supp } T$.*

(b) *The function $F(\xi, \eta) \equiv T_{f_{\xi, \eta}}$ is independent of the choice of $f_{\xi, \eta}$ in (a), and is a strongly continuous function of $(\xi, \eta) \in R \times C$.*

(c) *There is a number $C > 0$ and a positive integer N such that*

$$\|F(\xi, \eta)\| \leq C(1 + |\xi + i \text{Re } \eta|)^N (1 + |\text{Im } \eta|)^N$$

for all $(\xi, \eta) \in R \times C$.

PROOF. (a) Let $\varphi \in \mathcal{D}(R^1)$ be a test function which is identically 1 near the origin in R^1 . Define $f_{\xi, \eta}(x_1, x_2) = \exp[-i(x_1 \xi + \lambda_2 \eta)] \cdot \varphi(x_2)$ for all $(x_1, x_2) \in R^2$.

(b) The strong continuity is proved analogously to Theorem 2(b).

(c) With φ as above, define a mapping $u: \mathcal{E}(R^1) \rightarrow \mathcal{L}(X)$ as follows. For χ in the space $\mathcal{E}(R^1)$, let $\chi_\varphi(x_1, x_2) \equiv \chi(x_2) \cdot \varphi(x_2)$. Then $\chi_\varphi \in \mathcal{B}(R^2)$, so that we can define $u(\chi) \equiv T_{\chi_\varphi}$. The correspondence $\chi \rightarrow u(\chi)$ is clearly continuous with respect to the usual topology of $\mathcal{E}(R^1)$. Furthermore, if $0 \notin \text{supp } \chi$, then the real axis does not intersect $\text{supp } \chi_\varphi$, so $u(\chi) = T_{\chi_\varphi} = 0$. Thus $\text{supp } u \subset \{0\}$. Setting $t \equiv \text{Im } \eta$, define $\chi_0(x_2) \equiv (tx_2)^{q+1}$, where q is the order of u . Then χ_0 vanishes, with all derivatives of order $\leq q$, on $\{0\} \supset \text{supp } u$. By the operator-valued analogy of [2, Theorem 1.5.4] $T_{(\chi_0)_\varphi} = u(\chi_0) = 0$; i.e., $T_{(tx_2)^{q+1}\varphi} = 0$. Since $(tx_2)^{q+1}\varphi = (t\lambda_2 q)^{q+1}$ near the real axis ($x_2 = 0$), we have $(T_{tx_2\varphi})^{q+1} = T_{(tx_2\varphi)^{q+1}} = 0$, so that

$$\exp(T_{tx_2\varphi}) = \sum_{n=0}^q \frac{(T_{tx_2\varphi})^n}{n!}.$$

On the other hand, the series for $\exp(tx_2\varphi)$ converges in $\mathcal{B}(R^2)$, so $\exp(T_{tx_2\varphi}) = T_{\exp(tx_2\varphi)}$. But $T_{\exp(tx_2\varphi)} = F(0, it)$, so

$$\begin{aligned} (iii) \quad \|F(0, it)\| &= \|\exp(T_{tx_2\varphi})\| = \left\| \sum_{n=0}^q \frac{(T_{tx_2\varphi})^n}{n!} \right\| \\ &\leq \sum_{n=0}^q \frac{|t|^n}{n!} \|T_{tx_2\varphi}\|^n \leq C(1 + |t|)^q \end{aligned}$$

for all $t \in R^1$ and for some $C > 0$.

Now $F(\xi, \eta) = F(\xi, \operatorname{Re} \eta)F(0, i\eta)$, so the desired estimate follows from (iii) and Theorem 2(e).

The domain of the associated generalized scalar.

THEOREM 4. *Let A be a generalized scalar operator in X [6, Definition 3] with spectral distribution T . Then for a vector $x \in X$, x is in the domain D_A of A if and only if $(\partial/\partial\xi)F(\xi, \eta)x$ and $(\partial/\partial\eta)F(\xi, \eta)x$ exist for all $(\xi, \eta) \in R^2$. For $x \in D_A$, we have*

$$(iv) \quad Ax = i(\partial/\partial\xi)F(\xi, \eta)x|_{\xi=\eta=0} - (\partial/\partial\eta)F(\xi, \eta)x|_{\xi=\eta=0}.$$

To prove this theorem we need the following two lemmas.

LEMMA A. *For all $x \in D_A$, $\lim_{n \rightarrow \infty} T_{x_1\varphi_n}x$ and $\lim_{n \rightarrow \infty} T_{x_2\varphi_n}x$ exist, where $\{\varphi_n\}$ are as usual.*

LEMMA B. *Let $U, V: R^2 \rightarrow X$ be continuous mappings into the Banach space X . Assume*

$$-\int U \cdot (\partial/\partial x_1)\varphi \, dx_1 \, dx_2 = \int \varphi \cdot V \, dx_1 \, dx_2$$

for all $\varphi \in \mathcal{D}(R^2)$. Then $(\partial/\partial x_1)U$ exists for all $(x_1, x_2) \in R^2$, and equals V .

PROOF OF LEMMA A. The double sequence of functions $\psi_{nm} \equiv (x_1/(x_1 + ix_2))(\varphi_n - \varphi_m)$ is $\mathcal{B}(R^2)$ -bounded, and $\inf\{|\lambda|: \lambda \in \operatorname{supp} \psi_{nm}\} \rightarrow_{nm} \infty$. Thus $\|T_{x_1\varphi_n}x - T_{x_1\varphi_m}x\| = \|T_{\lambda\psi_{nm}}x\| \rightarrow_{nm} 0$. Similarly for the other limit. For obvious reasons, we define

$$(\operatorname{Re} A)x \equiv \lim_n T_{x_1\varphi_n}x, \quad (\operatorname{Im} A)x \equiv \lim_n T_{x_2\varphi_n}x.$$

These definitions are independent of the choice of $\{\varphi_n\}$.

The proof of Lemma B is analogous to [2, Theorem 1.4.2].

PROOF OF THEOREM 4. Assume $x \in D_A$. Then for any $\varphi \in \mathcal{D}(R^2)$, we have

$$-\int \frac{\partial}{\partial x_1} \varphi(x_1, x_2)F(x_1, x_2)x \, dx_1 \, dx_2 = \hat{T}_{-\partial\varphi/\partial x_1}x = T_{-(\partial\varphi)^{\wedge}/\partial x_1}x = T_{-i\xi\hat{\varphi}}x$$

(where $[\xi\hat{\varphi}](\xi, \eta) \equiv \xi\hat{\varphi}(\xi, \eta)$ for all $(\xi, \eta) \in R^2$)

$$= \lim_n T_{\varphi_n}T_{-i\xi\hat{\varphi}}x = \lim_n T_{-i\xi\varphi_n\hat{\varphi}}x = -i \lim_n T_{\varphi_n}T_{\xi\varphi_n}x$$

$$= -iT_{\varphi}(\operatorname{Re} A)x = -i \int \varphi(x_1, x_2)F(x_1, x_2)(\operatorname{Re} A)x \, dx_1 \, dx_2.$$

Lemma B implies

$$(v) \quad (\partial/\partial\xi)F(\xi, \eta)x = -iF(\xi, \eta)(\operatorname{Re} A)x \quad \text{for all } (\xi, \eta) \in R^2.$$

Similarly,

$$(vi) \quad (\partial/\partial\eta)F(\xi, \eta)x = -iF(\xi, \eta)(\text{Im } A)x \quad \text{for all } (\xi, \eta) \in R^2.$$

Results (v) and (vi) in turn imply

$$\begin{aligned} Ax &= (\text{Re } A)x + i(\text{Im } A)x = F(0, 0)(\text{Re } A)x + iF(0, 0)(\text{Im } A)x \\ &= i(\partial/\partial\xi)F(\xi, \eta)x|_{\xi=\eta=0} - (\partial/\partial\eta)F(\xi, \eta)x|_{\xi=\eta=0}. \end{aligned}$$

We will show that the existence of

$$(\partial/\partial\xi)F(\xi, \eta)x|_{\xi=\eta=0} \quad \text{and} \quad (\partial/\partial\eta)F(\xi, \eta)x|_{\xi=\eta=0}$$

implies $x \in D_A$. Let $y = (\partial/\partial\xi)F(\xi, \eta)x|_{\xi=\eta=0}$. Then $(\partial/\partial\xi)F(\xi, \eta)x = F(\xi, \eta)y$, and for $\varphi \in \mathcal{D}(R^2)$ we have

$$\begin{aligned} T_\varphi y &= \hat{T}_\psi y \quad (\text{where } \psi \in \mathcal{S}(R^2), \hat{\psi} = \varphi) \\ &= \int \psi(\xi, \eta)F(\xi, \eta)y \, d\xi \, d\eta = \int \psi(\xi, \eta)(\partial/\partial\xi)F(\xi, \eta)x \, d\xi \, d\eta \\ &= - \int (\partial/\partial\xi)\psi(\xi, \eta)F(\xi, \eta)x \, d\xi \, d\eta = \hat{T}_{-\partial\psi/\partial\xi}x \\ &= T_{-(\partial\psi) \wedge / \partial\xi}x = T_{-ix_1 \hat{\psi}}x = -iT_{x_1 \varphi}x. \end{aligned}$$

Similarly, if $z = (\partial/\partial\eta)F(\xi, \eta)x|_{\xi=\eta=0}$ then $T_\varphi z = -iT_{x_2 \varphi}x$. Thus $T_{\lambda \varphi}x = iT_\varphi y - T_\varphi z$. Now the vanishing of T to power 0 at infinity implies T vanishes to power 1 at infinity for x : i.e., $x \in D_A$.

REMARK. If $\text{supp } T$ is real, then the function $g_\xi(t) \equiv F(\xi, it)$ is differentiable with respect to $t \in R$, for any $\xi \in R$, in the uniform operator topology. This follows from the representation $F(0, it) = T_{\exp(itx_2 \varphi)}$ for φ a test function identically 1 near the origin in R^1 . (Cf. proof of Theorem 3(c).)

Generalized pseudo-hermitian operators.

DEFINITION. A mapping A of a Banach space X into itself is called generalized pseudo-hermitian (g.p.h.) if

(a) A is a (not necessarily bounded) generalized scalar [6, Definition 3], and

(b) The spectrum $\sigma(A)$ of A is real.

REMARK. [6, Theorem 4] implies that $\text{supp } T$ is real for any spectral distribution T which is admitted by a g.p.h. operator A .

DEFINITION. A group of operators $\{U_t\}_{t \in R}$ in a Banach space X is called temperate if there are a number $C > 0$ and a positive integer N such that $\|U_t\| \leq C(1 + |t|)^N$ for all $t \in R$.

THEOREM 5. *Let A be a g.p.h. operator in X . Then $-i \cdot A$ is the infinitesimal generator of a temperate C_0 group of operators in X .*

PROOF. Let T be a spectral distribution which A admits. Then $\text{supp } T$ is real. Let $F(\xi, \eta)$ be as in Theorem 3. Define $U_t \equiv F(t, it)$. By Theorem 2(c), $U_0 = I$. By Theorem 3(b), U_t is strongly continuous. The group property $U_{t+s} = U_t U_s$ is clear. By Theorem 3(c), $\|U_t\| \leq C(1+|t|)^{2N}$, so U_t is a temperate C_0 group. Let B denote the infinitesimal generator of U_t . If $x \in D_A$, then Theorem 4 (together with the following remark) implies the existence of $(d/dt)(U_t x)$:

$$\begin{aligned} (d/dt)(U_t x)|_{t=0} &= (\partial/\partial\xi)F(\xi, \eta)x|_{\xi=\eta=0} + (d/dt)[F(0, it)x]|_{t=0} \\ &= -i(\text{Re } A)x + (d/dt)[T_{\exp(itx_2\varphi)}x]|_{t=0} \\ &= -i(\text{Re } A)x + T_{x_2\varphi}x \\ &= -i(\text{Re } A)x + (\text{Im } A)x = -i \cdot Ax. \end{aligned}$$

Thus $x \in D_B$ and $Bx = -i \cdot Ax$.

On the other hand, let $x \in D_B$. Then the existence of the derivative $(\partial/\partial\xi)F(\xi, \eta)x|_{\xi=\eta=0}$ follows from the relation

$$F(\xi, 0)x = F(0, -i\xi)F(\xi, i\xi)x,$$

remembering the remark after Theorem 4 and the differentiability of $F(\xi, i\xi)x = U_\xi x$ with respect to ξ . The derivative $(\partial/\partial\eta)F(0, \eta)$ exists in the uniform operator topology since $\text{supp } T$ real implies that $F(0, \eta) = T_{\exp(-ix_2\eta\varphi)}$. Thus Theorem 4 implies $x \in D_A$, and we have $B = -i \cdot A$.

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