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ON AN INTEGRAL FORMULA FOR CLOSED HYPERSURFACES OF THE SPHERE

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ABSTRACT. In a compact oriented hypersurface M^n of the sphere S^{n+1} the integral formula $\int_{M^n} \nabla K_r\, dV = n \int_{M^n} (K_r K_1 - K_{r+1}) e\, dV$ is proved where K_r is the rth mean curvature, e is the unit normal of M^n in S^{n+1}. Some applications are considered.

1. Let S^{n+1} be the unit sphere in a euclidean space E^{n+2} and $x: M^n \to S^{n+1}$ be an isometric immersion of a compact oriented Riemannian manifold M^n of dimension n into S^{n+1}. Let $F(M^n)$, $F(S^{n+1})$ and $FE(^{n+2})$ be the bundles of orthonormal frames of M^n, S^{n+1} and E^{n+2} respectively. Let B be the set of elements $b = (p, e_1, \cdots, e_n, e, x(p))$ such that $(p, e_1, \cdots, e_n) \in F(M^n)$, $(x(p), e_1, \cdots, e_n, e) \in F(S^{n+1})$ and $(x(p), e_1, \cdots, e_n, e, x) \in F(E^{n+2})$ with coherent orientations. $dx(e_i)$ is identified with e_i, $i = 1, 2, \cdots, n$. Define $\tilde{x}: B \to F(E^{n+2})$ by $\tilde{x}(b) = (x(p), e_1, \cdots, e_n, e, x)$.

By the structure equations of E^{n+2} and the pullback by $\tilde{x}$ we may write

$$dx = \sum \omega_i e_i, \qquad de = \sum_i \theta_i e_i \tag{1.1}$$

with $\theta_i = k_i w_i$. Where $i = 1, 2, \cdots, n$; $k_1, \cdots, k_n$ are principal curvatures of M^n in S^{n+1} at p. de does not have component in the x direction is easily followed from $d(e \cdot x) = 0$.

2. Let $|\cdot, \cdots, \cdot|$ denote the combined operation of the vector product and exterior product ([1], [3], [4]). Put

$$\Delta_r = |\underbrace{de, \cdots, de}_{(r\text{ times})}, \underbrace{dx, \cdots, dx}_{(n-r-1\text{ times})}, e, x|. \tag{2.1}$$

Then

$$(-1)^{n-1} d\Delta_r = |\underbrace{de, \cdots, de}_{(r+1\text{ times})}, \underbrace{dx, \cdots dx}_{(n-r-1\text{ times})}, x| - |\underbrace{de, \cdots, de}_{(r\text{ times})}, \underbrace{dx, \cdots, dx}_{(n-r\text{ times})}, e|$$

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Using (1.1) and straight computation we have

$$|\underbrace{de, \cdots, de}_{(r+1 \text{ times})}, \underbrace{dx, \cdots, dx}_{(n-r-1 \text{ times})}, x| = -n!\, K_{r+1} e\, dV$$

and

$$|\underbrace{de, \cdots, de}_{(r \text{ times})}, \underbrace{dx, \cdots, dx}_{(n-r \text{ times})}, e| = n!\, K_r x\, dV$$

where $dV = \omega_1 \wedge \cdots \wedge \omega_n$ is the volume element in M^n and K_r is the rth mean curvature of M^n in S^{n+1} defined by the elementary symmetric functions of $k_1, \cdots, k_n$ as follows:

$$\binom{n}{r} K_r = \sum_{j_1 < \cdots < j_r} k_{j_1} k_{j_2} \cdots k_{j_r} \qquad (1 \leqq r \leqq n).$$

Thus

$$d\Delta_r = (-1)^n n!\, (K_{r+1} e + K_r x)\, dV, \tag{2.2}$$

and by Stokes' theorem we have

$$\int_{M^n} (K_{r+1} e + K_r x)\, dV = 0, \qquad r = 1, 2, \cdots, n-1. \tag{2.3}$$

This integral formula (2.3) has been obtained by Reilly [5].

3. Substituting (1.1) in the right side of (2.1) yields [1]

$$\Delta_r = (-1)^{n+1} r!\, (n-r-1)! \sum_{i=0}^{r} (-1)^i \binom{n}{r-i} K_{r-i} {}^*U_i \tag{3.1}$$

where * is the Hodge star operation and

$$U_i \overset{\text{def}}{=} \sum_j (k_j)^i \omega_j e_j,$$

$${}^*U_i = \sum_j (-1)^{j-1} (k_j)^i \omega_1 \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_n e_j,$$

$i = 0, 1, \cdots, n$. Using (3.1) to calculate $d\Delta_r$ we have

$$d\Delta_r = (-1)^{n+1} r!\, (n-r-1)! \left[\binom{n}{r} dK_r \wedge {}^*dx + \binom{n}{r} K_r\, d({}^*dx) + \sum_{i=1}^{r} (-1)^i \binom{n}{r-i} d(K_{r-i} {}^*U_i) \right].$$

It is easy to show that

$$dK_r \wedge {}^*dx = \nabla K_r\, dV,$$
$$d({}^*dx) = -n(K_1 e + x)\, dV.$$

Hence we have

$$(3.2)\qquad \begin{aligned} d\Delta_r &= (-1)^{n+1} r!\,(n-r-1)! \\ &\cdot \left[\binom{n}{r} \nabla K_r\, dV - n\binom{n}{r} K_1 K_r e\, dV \right. \\ &\left. - n\binom{n}{r} K_r x\, dV + \sum_{i=1}^{r} (-1)^i \binom{n}{r-i}\, d(K_{r-i} {}^*U_i) \right]. \end{aligned}$$

On the other hand by (3.1) and (2.2) we obtain

$$dx \cdot \Delta_r = (-1)^{n+1} r!\,(n-r-1)! \sum_{i=0}^{r} \left[(-1)^i \binom{n}{r-i} K_{r-i} \sum_j (k_j)^i \right] dV,$$

$$x \cdot d\Delta_r = (-1)^n n!\, K_r\, dV.$$

From $x \cdot \Delta_r = 0$ it follows $0 = d(x \cdot \Delta_r) = dx \cdot \Delta_r + x\, d\Delta_r$ and hence

$$(3.3)\qquad r\binom{n}{r} K_r - \sum_{i=1}^{r} (-1)^i \binom{n}{r-i} K_{r-i} \sum_j (k_j)^i = 0.$$

This is the well-known identity of Newton.

Since $d(K_r {}^*dx) = \nabla K_r\, dV - nK_r(K_1 e + x)\, dV$, one obtains by the Stokes' theorem

$$\int_{M^n} [\nabla K_r - nK_r(K_1 e + x)]\, dV = 0.$$

Together with (2.3) we have the following theorem.

THEOREM. *Let M^n be a compact oriented hypersurface in S^{n+1}, K_r be the rth mean curvature of M^n in S^{n+1}, e be the unit normal of M^n in S^{n+1}. Then*

$$(3.4)\qquad \int_{M^n} \nabla K_r\, dV = n \int_{M^n} (K_r K_1 - K_{r+1}) e\, dV.$$

4. The following are some applications of the theorem.

COROLLARY 1. *In the theorem suppose, furthermore, that there is a fixed vector a in E^{n+2} such that the function $a \cdot e$ is of the same sign on M^n, $K_i > 0$ for $i = 1, \cdots, r$, $1 \leqq r \leqq n-1$, and K_r is constant. Then M^n is a hypersphere in S^{n+1}.*

PROOF. Under the assumption we have that $K_r K_1 - K_{r+1} = 0$. The same argument as in [**4**, p. 731] yields $k_1 = k_2 = \cdots = k_n$ at all points of M^n. Hence M^n is a hypersphere in S^{n+1}.

COROLLARY 2. *In the theorem suppose, furthermore, that M^n is minimal in S^{n+1} and that there is a fixed vector a in E^{n+2} such that the function $a \cdot e$ is of the same sign on M^n. Then M^n is totally geodesic.*

PROOF. By the assumptions and (3.4) it implies that $K_1=K_2=0$. So $k_i=0$ ($i=1, 2, \cdots, n$) and M^n is totally geodesic. This result is known [**2**, p. 33].

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