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ON CLOSED CURVES IN MINKOWSKI SPACES

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ABSTRACT. The minimum pseudo-diameter d and the length L of a simple closed rectifiable curve in Minkowski space satisfy $L \geqq gd$ where g is the half-girth of the unit ball. The bound is sharp.

This note answers some of the questions raised by H. Herda in **[1]**.

Let M be a Minkowski (finite dimensional real Banach) space with unit ball B. Let γ be a simple closed rectifiable curve in M. For x on γ let x' be the point whose distance along γ to x is half the length $L(\gamma)$ of the curve γ. Let $d(\gamma)=\min_{x\in\gamma}\|x-x'\|$ and let g be the half-girth **[2]** of B, that is, the shortest length of a curve on ∂B with antipodal endpoints.

THEOREM 1. *One has $L(\gamma)\geqq gd(\gamma)$ for all γ and there exists a curve for which equality holds.*

PROOF. Let γ be given. Choose a point a_1 and an orientation on γ. For arbitrary $\varepsilon \in (0, \frac{1}{2})$ choose an integer $k>0$ such that $2^{-k}L(\gamma)<\varepsilon d(\gamma)$. Define points $a_i \in \gamma$, $i=1, \cdots, 2^k$, succeeding each other in the chosen direction with the γ-arc from a_i to a_{i+1} of length $2^{-k}L(\gamma)$. Then $a_i'=a_j$ where $j\equiv i+2^{k-1} \bmod 2^k$. Join the a_i by straight line segments in index order to form a closed polygon α. Let $r_i=\frac{1}{2}(a_i-a_i')$, $m_i=\frac{1}{2}(a_i+a_i')$.

Join the points r_i in index order to form a closed polygon ρ, centrally symmetric about the origin. By convexity of the norm one has

$$\begin{aligned}\|a_{i+1}-a_i\|+\|a'_{i+1}-a'_i\| &= \|(m_{i+1}-m_i)+(r_{i+1}-r_i)\| \\ &\quad + \|(m_{i+1}-m_i)-(r_{i+1}-r_i)\| \\ &\geqq 2\,\|r_{i+1}-r_i\|.\end{aligned}$$

Since $\|a_{i+1}-a_i\|\leqq 2^{-k}L(\gamma)$ for all i, one has

$$\|r_{i+1}-r_i\| \leqq 2^{-k}L(\gamma) < \varepsilon d(\gamma).$$

By construction $\|r_i\|=\frac{1}{2}\|a_i-a_i'\|\geqq\frac{1}{2}d(\gamma)$. All vertices of polygon ρ are therefore on the boundary or exterior of the ball $\frac{1}{2}d(\gamma)B$. Then no point

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of ρ can lie in the interior of the ball $(\frac{1}{2}-\varepsilon)d(\gamma)B$. Therefore

$$L(\gamma) \geqq L(\alpha) = \sum_{i=1}^{2^{k-1}} (\|a_{i+1} - a_i\| + \|a'_{i+1} - a'_i\|)$$
$$\geqq 2\sum_{i=1}^{2^{k-1}} \|r_{i+1} - r_i\| = L(\rho) \geqq (1 - 2\varepsilon)gd(\gamma)$$

where g is the half-girth of B, the last inequality by Lemmas 3.2 and 5.1(a) of Schäffer [2]. This proves that the inequality holds. By Lemma 5.1(b) of [2] there exists a curve γ on ∂B, centrally symmetric about the origin, for which $L(\gamma)=2g$ and, trivially, $d(\gamma)=2$, yielding equality.

REMARKS. 1. In infinite dimension the same argument applies but a curve yielding equality may not exist.

2. In 2 dimensions, $L(\gamma)=gd(\gamma)$ does not imply that γ is a homothet of ∂B. When the unit ball is a square, an infinity of curves exists which have $L(\gamma)=8$, $d(\gamma)=2$ and up to twice the area of B.

3. For Euclidean and Hilbert spaces R. Ault has shown [4] that $L=\pi d$ only for circles, settling Herda's conjecture in the affirmative. An earlier partial proof for E^2 was provided by A. M. Fink [3].

4. Herda's second conjecture states that for a simple closed rectifiable curve γ in E^2 the following are equivalent:

(i) the curve admits a unique tangent at each point;

(ii)
$$\inf_{t\in(0,d)} \max_{x\in\gamma} a(x, t)/t = 1$$

where $a(x, t)$ is the shortest arc length from point x, in a fixed orientation along γ, to a point of γ at Euclidean distance t from x. This conjecture is false. Let γ_1 be the graph in E^2 of the function f defined on $[-1, +1]$ by $f(0)=0, f(x)=x^2 \sin x^{-1}$. Complete γ_1 into a simple closed curve γ by a smooth connection of its endpoints. Then γ is rectifiable and has a unique tangent at each point while (ii) fails because

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\varepsilon (1 + \cos^2 t^{-1})^{1/2}\, dt > 1.$$

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