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THE TOPOLOGICAL COMPLEMENTATION THEOREM À LA ZORN

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ABSTRACT. Steiner's topological complementation theorem is given a short simple proof using Zorn's Lemma.

A. K. Steiner [3, Theorem 7.8, p. 397] proved that the lattice of topologies on a fixed set X, denoted Σ or $\Sigma(X)$, is complemented.[1] In fact, she showed that each $t \in \Sigma$ has a complement in $\Pi = \Pi(X)$, the sublattice of principal topologies. (A topology $t \in \Sigma$ is *principal* iff each point $x \in X$ has a smallest t-neighborhood: $B_t(x)$.) Her proof was quite complicated and although van Rooij [1] gave a simpler proof, his proof used both Zorn's Lemma and two applications of transfinite induction. The purpose of this note is to prove Steiner's Theorem via a standard Zornification by the simple trick of suitably adjoining a new point p to X and subsequently discarding it.

THEOREM (A. K. STEINER). *Every $t \in \Sigma$ has a complement $t' \in \Pi$.*

PROOF. Take a point $p \notin X$ and let T be the topology defined on $Y = X \cup \{p\}$ by $T = t \cup \{U \cup \{p\} \mid U \in t\}$. Let $\mathscr{A} = \{(A, s) \mid p \in A \subset Y$ and $s \in \Pi(A)$ is a complement for $T|A\}$. Then $\mathscr{A} \neq \varnothing$, since $(\{p\}, \{\varnothing, \{p\}\}) \in \mathscr{A}$. Partially order $\mathscr{A}$ by $(A_1, s_1) \leqq (A_2, s_2)$ iff (i) $A_1 \subset A_2$, (ii) $B_1(x) = B_2(x)$ for $x \in A_1 \backslash \{p\}$, and (iii) $B_1(p) \subset B_2(p) \subset B_1(p) \cup A_2 \backslash A_1$. If

$$\mathscr{B} = \{(A_i, s_i) \mid i \in I\} \subset \mathscr{A}$$

is totally ordered, let (A, s) be defined by $A = \bigcup A_i$; $B_s(x) = B_i(x)$ if $x \in A_i \backslash \{p\}$ and $B_s(p) = \bigcup B_i(p)$. It is easily verified that (A, s) is an upper bound for $\mathscr{B}$ in $\mathscr{A}$, so by Zorn's Lemma $\mathscr{A}$ has a maximal element, say (M, m). But, $M = Y$. For otherwise, if $q \in Y \backslash M$ we can extend m to m' on

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[1] For $t_1, t_2 \in \Sigma$ we have $t_1 \leqq t_2$ iff $t_1 \subset t_2$. We say that $t' \in \Sigma$ is a *complement* for $t \in \Sigma$ iff $t \vee t' = 1$, the discrete topology, and $t \wedge t' = 0$, the trivial topology. See [2] for the cardinality of the set of complements.

$M' = M \cup \{q\}$ by: $B_{m'}(p) = B_m(p)$ if M is not open in M', $B_{m'}(p) = B_m(p) \cup \{q\}$ if M is open in M', $B_{m'}(q) = \{q\}$ if $\{q\}$ is not open in M', $B_{m'}(q) = B_m(p) \cup \{q\}$ if $\{q\}$ is open in M'. Since $(M, m) \leqq (M', m') \in \mathscr{A}$, this is a contradiction. It immediately follows that, since m is a principal complement for T and since both X and $\{p\}$ are T-open, $m|X$ is a principal complement for t. $\square$

References

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3. A. K. Steiner, *The lattice of topologies: Structure and complementation*, Trans. Amer. Math. Soc. **122** (1966), 379–398. MR **32** #8303.

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