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SIMPLY-CONNECTED BRANCHED COVERINGS OF S^3

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ABSTRACT. It is shown that a simply connected covering of S^3 branched over a torus knot or one of a certain class of links is S^3.

A possible approach to the 3-dimensional Poincaré conjecture is outlined by Fox in [**5**]: Construct a compact simply-connected covering of S^3 branched over a tame link, and then try to decide whether or not it is in fact S^3. It is the purpose of the present note to point out that if the complement of the link in question is a Seifert fibre space, then this construction will never yield a counterexample to the Poincaré conjecture. In particular, the simply-connected branched covering over the trefoil described in detail in [**4**] and [**5**] is indeed S^3. (This last fact also follows from a theorem of Burde [**1**].)

LEMMA. *Let M be a Seifert fibre space* (*with or without boundary*), *and suppose that $K_1, \cdots, K_n \subset \operatorname{int} M$ are fibres* (*ordinary or exceptional*). *Then if $\tilde{M}$ is any compact covering of M branched over $K_1 \cup \cdots \cup K_n$, $\tilde{M}$ is also a Seifert fibre space.*

PROOF. Removing the interior of a tubular neighbourhood T_i of each K_i gives a Seifert fibre space N. If $\tilde{N}$ is that part of $\tilde{M}$ which lies over N, then $\tilde{N}$ is a Seifert fibre space, and the covering projection $p:\tilde{N}\to N$ takes fibres to fibres [**6**, p. 195]. The branched covering $\tilde{M}$ is then obtained by attaching solid tori $\tilde{T}_{i,j}$, $j=1, \cdots, k_i$, $i=1, \cdots, n$, to $\tilde{N}$, in such a way that a meridian on $\partial\tilde{T}_{i,j} \subset \partial\tilde{N}$ projects, under p, to a meridian on $\partial T_i \subset \partial N$ [**6**, pp. 231–233]. Since the fibring of N extends to a fibring of M, the fibres on ∂T_i are not meridians. Therefore the fibres on $\partial\tilde{T}_{i,j}$ are not meridians, and the fibring of $\tilde{N}$ can be extended to a fibring of $\tilde{M}$.

THEOREM. *If $\tilde{M}$ is a compact simply-connected covering of S^3 branched over any link whose complement is a Seifert fibre space, then $\tilde{M} \simeq S^3$.*

REMARK. A complete description of such links is given in [**2**]. In particular, the knots with this property are precisely the torus knots (see also [**3**]).

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PROOF OF THEOREM. If the fibring of the complement of the link extends to a fibring of S^3 in which the components of the link are fibres, then $\tilde{M}$ is a Seifert fibre space. Since it is simply-connected, it is therefore S^3 [**6**, p. 206].

If the fibring of the complement does not so extend, it can at least be extended to all components of the link except one, say K [**2**]. If T is a tubular neighbourhood of K, and Q the complement of the interior of T, then Q is a Seifert fibre space such that the fibres on $\partial T = \partial Q$ are meridians. Then $\tilde{M} = (\tilde{T}_1 \cup \cdots \cup \tilde{T}_k) \cup \tilde{Q}$, where $\tilde{T}_1, \cdots, \tilde{T}_k$ are solid tori, and $\tilde{Q}$ is a Seifert fibre space such that the fibres on $\partial \tilde{T}_j \subset \partial \tilde{Q}$, $j = 1, \cdots, k$, are meridians. Now $\pi_1(\tilde{M}) \simeq \pi_1(\tilde{Q})/\langle h \rangle$, where h is the element represented by an ordinary fibre of $\tilde{Q}$, and by hypothesis this is the trivial group. The argument in [**2**, p. 90] then shows that in fact $k=1$, the Zerlegungsfläche is a disc, and there are no exceptional fibres. Hence $\tilde{Q} \simeq D^2 \times S^1$, and $\tilde{M} \simeq (S^1 \times D^2) \cup (D^2 \times S^1)$, identified along $S^1 \times S^1$ by the identity, which is just S^3.

REFERENCES

1. G. Burde, *On branched coverings of* S^3, Canad. J. Math. **23** (1971), 84–89.

2. G. Burde and K. Murasugi, *Links and Seifert fiber spaces*, Duke Math. J. **37** (1970), 89–93. MR **40** #6528.

3. G. Burde and H. Zieschang, *Eine Kennzeichnung der Torusknoten*, Math. Ann. **167** (1966), 169–176. MR **35** #1008.

4. R. H. Fox, *Free differential calculus*, III. *Sugbroups*, Ann. of Math. (2) **64** (1956), 407–419. MR **20** #2374.

5. ———, *Construction of simply connected* 3*-manifolds*, Topology of 3-Manifolds and Related Topics (Proc. The Univ. of Georgia Inst., 1961), Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 213–216. MR **25** #3539.

6. H. Seifert, *Topologie dreidimensionaler gefaserter Räume*, Acta Math. **60** (1933), 147–238.

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