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ERROR BOUNDS FOR GALERKIN'S METHOD FOR MONOTONE OPERATOR EQUATIONS

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ABSTRACT. An abstract theorem, generalizing a result of Nitsche, is proved. This gives sharp error bounds for the Galerkin method for approximating the solutions of a large class of nonlinear operator equations in Hilbert spaces.

Let H be a real Hilbert space and T be a strongly monotone operator on H in the sense of Browder, i.e.,

$$|(Tu - Tv, u - v)_H| \geqq \gamma \, \|u - v\|_H^2 \tag{1}$$

for all $u, v \in H$ and some constant $\gamma > 0$. We are interested in numerically approximating the solution of the problem of finding $u \in H$ such that

$$Tu = f, \quad \text{where } f \text{ is given in } H, \tag{2}$$

by the Galerkin method. Given a finite-dimensional subspace, S, of H, the Galerkin method is to find $u_S \in S$ such that

$$(Tu_S, y)_H = (f, y)_H, \quad \text{for all } y \in S. \tag{3}$$

From [1] and [2], we recall the following result.

THEOREM 1. *If T is uniformly Lipschitz continuous for bounded arguments, i.e., given $B > 0$, there exists a positive constant, $C(B)$, depending only on B, such that $\|Tw - Tv\|_H \leqq C(B)\|w - v\|_H$ for all $w, v \in H$ with $\|w\|_H \leqq B$ and $\|v\|_H \leqq B$, then problems* (2) *and* (3) *have unique solutions and*

$$\|u - u_S\|_H \leqq \gamma^{-1} C(\|f - T0\|_H) \inf_{y \in S} \|u - y\|_H. \tag{4}$$

In many applications, H is a closed subspace of $W^{m,2}(\Omega)$ for some $m \geqq 1$ and (4) yields an error bound in the $W^{m,2}$-norm when we are really interested in an error bound in the L^2-norm. While the bound in (4) does induce an error bound in the L^2-norm, one might expect that such a bound is *not* sharp and indeed that is the case. In this note, we present an

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abstract theorem, generalizing a technique of Nitsche, cf. [3] and [4] for linear selfadjoint problems, which when applied to the problems in question yields sharp L^2-error bounds directly. See [5] for another such generalization.

Let V and W be two real Hilbert spaces such that $V \subset H \subset W$ and there exists a positive constant, K, such that

$$\|h\|_W \leqq K \|h\|_H, \quad \text{for all } h \in H. \tag{5}$$

As a concrete example, one may take $H \equiv W_0^{m,2}(\Omega)$, $V \equiv W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega)$, and $W \equiv L^2(\Omega)$.

Instead of (2), we consider the problem of finding $u \in H$ such that

$$(Tu, \phi)_H = (g, \phi)_W, \quad \text{for all } \phi \in H, \tag{6}$$

where g is given in W. Because of (5), problem (6) is a special case of problem (3).

Our new result is

THEOREM 2. *Let C be a collection of finite-dimensional subspaces, S, of H such that if u_S denotes the Galerkin approximation to u in S, then there exist $0 < \lambda \leqq \Lambda$ independent of S in C and a bilinear form b_S on H such that*

(i) *$(Tu - Tu_S, \phi)_H = b_S(u - u_S, \phi)$ for all $\phi \in H$ and all $S \in C$,*

(ii) *$b_S(\phi, \phi) \geqq \lambda \|\phi\|_H^2$ for all $\phi \in H$,*

(iii) *$|b_S(w, v)| \leqq \Lambda \|w\|_H \|v\|_H$ for all $w, v \in H$,*

(iv) *if $b_S(w, \phi_S) = (g, w)_W$ for all $w \in H$, then there exists a positive constant, ρ, independent of S in C, such that $\|\phi\|_V \leqq \rho \|g\|_W$, and*

(v) *there exists a positive function, E, on S such that*

$$\inf_{y \in S} \|g - y\|_H \leqq E(S) \|g\|_V, \quad \textit{for all } S \in C \textit{ and all } g \in V. \tag{7}$$

Then

$$\|u - u_S\|_W \leqq \gamma^{-1} \rho C^2(\|f - T0\|_H) E(S) \inf_{y \in S} \|u - y\|_H.$$

PROOF. For each $S \in C$, let $e_S \equiv u - u_S$ and consider the problem of finding $\phi_S \in H$ such that

$$b_S(w, \phi_S) = (e_S / \|e_S\|_W, w)_W, \quad \text{for all } w \in H. \tag{8}$$

By our hypotheses on b_S, this problem has a unique solution, ϕ_S, and $\|\phi_S\|_V \leqq \rho$.

Setting $w = e_S$, we have $\|e_S\|_W = b_S(e_S, \phi_S) = (Tu - Tu_S, \phi_S)$. Moreover, by the definition of the Galerkin method, we have

$$\|e_S\|_W = (Tu - Tu_S, \phi_S - y)_H, \quad \text{for all } y \in S.$$

Thus,

$$\|e_S\|_W \leqq C(\|f - T0\|_H) \|u - u_S\|_H \|\phi_S - y\|_H, \quad \text{for all } y \in S.$$

Using Theorem 1 to bound $\|u-u_S\|_H$ and (7), we obtain the required result. Q.E.D.

The reader is referred to [6] for further details and applications of this result to boundary value problems for linear and semilinear elliptic partial differential equations and eigenvalue problems.

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