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WEAK PARTITION RELATIONS

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ABSTRACT. The partition relation $\aleph_1 \to (\aleph_1)^2_{3,2}$, which was known to contradict the continuum hypothesis [1], is disproved without this hypothesis.

For cardinals $\kappa \geqq \omega$ and $\lambda \geqq 1$, let $P(\kappa, \lambda)$ be the following partition relation: For any mapping F of the set $[\kappa]^2$ of unordered pairs of elements of κ into λ, there is a set $B \subseteq \kappa$, of cardinality $|B| = \kappa$, such that the image of $[B]^2$ under F is not all of λ. Such a set B will be called *slightly homogeneous* for the partition F. Partition relations of this sort were studied by Erdös, Hájnal, and Rádo [**1**, especially §18], who assumed, in most of their theorems, that the continuum hypothesis holds. We shall show that $P(\aleph_1, 3)$ can be disproved outright (in Zermelo-Frankel set theory ZF with the axiom of choice).

Notice that, for $\lambda \leqq \mu$, $P(\kappa, \lambda)$ implies $P(\kappa, \mu)$. For, given a map $F: [\kappa]^2 \to \mu$, we let $G: [\kappa]^2 \to \lambda$ be the composite of F and a surjection $\mu \to \lambda$ and observe that any set which is slightly homogeneous for G is also slightly homogeneous for F. Thus, for a fixed κ, the partition relation $P(\kappa, \lambda)$ becomes weaker as λ increases. Using this fact, we easily obtain the following lemma by induction on m.

LEMMA 1. *If $n \leqq m < \omega$ and if $P(\kappa, n)$, then, for any map $F: [\kappa]^2 \to m$, there is a $B \subseteq \kappa$ with $|B| = \kappa$ and $|F([B]^2)| < n$.*

When $\lambda > \kappa$, then $P(\kappa, \lambda)$ is trivially true, because κ itself is slightly homogeneous for any F. At the other extreme, $P(\kappa, 1)$ is trivially false.

For finite λ, $P(\kappa, \lambda)$ coincides with the partition relation $\kappa \to (\kappa)^2_{\lambda, \lambda-1}$ of Kleinberg [**3**]. In particular, $P(\kappa, 2)$ holds if and only if κ is (strongly) inaccessible and weakly compact [**5**, Theorems 8.3 and 9.4], so it is relatively consistent with ZF to assume that $P(\kappa, 2)$ holds only for $\kappa = \omega$. ($P(\omega, 2)$ is a form of Ramsey's theorem [**6**].) On the other hand, one can prove the existence of uncountable κ such that $P(\kappa, 3)$. Indeed, for singular κ, $P(\kappa, 3)$ holds if and only if κ is a strong limit cardinal and $P(cf(\kappa), 2)$, so, for example, $P(\beth_\omega, 3)$ holds.

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If we assume the generalized continuum hypothesis, then the question whether $P(\kappa, \lambda)$ holds or not is completely answered for successor cardinals by the following result of Erdös, Hájnal, and Rádo [**1**, Theorem 17].

THEOREM 0. *If* $2^\kappa = \kappa^+$, *then not* $P(\kappa^+, \kappa^+)$.

COROLLARY. *If* $2^\kappa = \kappa^+$, *then* $P(\kappa^+, \lambda)$ *if and only if* $\lambda > \kappa^+$.

We shall prove the following theorem, which becomes a special case of the above corollary if one assumes the continuum hypothesis.

THEOREM 1. *Not* $P(\aleph_1, 3)$.

This theorem has subsequently been improved by Galvin and Shelah [**2**] who have shown that $P(\aleph_1, 4)$ is also false. Their proof is quite similar to my proof of Theorem 1; in fact both proofs use the same partition. As far as I know, the consistency of $P(\aleph_1, 5)$ is still an open question.

The first (and longest) part of the proof of Theorem 1 will be the construction of three linearly ordered sets, each of cardinality $\aleph_1$, such that no uncountable subset of any of the three is isomorphic or anti-isomorphic to a subset of any other. One of the three sets will be $\aleph_1$ with its standard ordering (as an ordinal). Before describing the others, we introduce some terminology.

DEFINITION. A subset X of a linearly ordered set Y is *pseudo-dense* in Y if it meets every half-open interval $[a, b)$ of Y.

LEMMA 2. *There is a linearly ordered set A of cardinality $\aleph_1$ such that*

(a) *There are no monotone (increasing or decreasing) sequences of length $\aleph_1$ in A.*

(b) *For every subset Y of A there is a countable subset X of Y which is pseudo-dense in Y.*

(c) *A is isomorphic to its dual A^*.*

PROOF. Notice that any subset Y of the real line has the property required in (b). The required X is obtained by choosing one element from each nonempty intersection $Y \cap (p, q)$ of Y with a rational interval, subject to the restriction that, if such an intersection has a largest element, then that largest element is to be the chosen one. If $[a, b)$ is an interval of Y and there is a $c \in Y$ strictly between a and b, then, for rationals p, q such that $a<p<c<q<b$, the chosen element of $Y \cap (p, q)$ is in $[a, b) \cap X$. On the other hand, if no such c exists, then, for rationals p, q such that $p<a<q<b$, the intersection $Y \cap (p, q)$ has a largest element, namely a, so $a \in [a, b) \cap X$. Thus, X is pseudo-dense in Y.

It follows that any symmetric subset A of the reals satisfies (b) and (c). Property (a) follows from (b) and (c) since $\aleph_1$ has no proper pseudo-dense subset. As we can obviously choose such an A of cardinality $\aleph_1$, the lemma is proved. □

The A provided by Lemma 2 will be the second of our three sets. To describe the third, we shall need some facts about trees.

A tree is a partially ordered set $(T, <)$ such that, for each $x \in T$, the set $\{y \mid y<x\}$ of strict predecessors of x is well-ordered. The height $h(x)$ of x is the order type of $\{y \mid y<x\}$, and the height $h(T)$ of T is the least ordinal not of the form $h(x)$ for any $x \in T$. If $\alpha \leqq h(x)$, then $p_\alpha(x)$ is the (unique) predecessor of x of height α, the αth element of $\{y \mid y \leqq x\}$. The αth level of T is $\{x \mid h(x)=\alpha\}$. A path through T is a linearly ordered subset of T which meets the αth level for all $\alpha<h(T)$.

Suppose $(T, <)$ is a tree and $<'$ is a linear ordering of T (which may be totally unrelated to the tree ordering $<$). If x and y are incomparable elements of T (with respect to $<$) and α is the smaller of their heights, then clearly $p_\alpha(x) \neq p_\alpha(y)$. Let β be the least ordinal such that $p_\beta(x) \neq p_\beta(y)$. We shall say that x is *left* of y (and y is *right* of x) if $p_\beta(x) <' p_\beta(y)$. We define a relation $\prec$ on T by

$$x \prec y \quad \text{iff} \quad x < y \text{ or } x \text{ is left of } y.$$

The straightforward proof of the following lemma will be left to the reader.

LEMMA 3. *With the notations introduced above, $\prec$ linearly orders T.* □

We shall need the following result of Aronszajn; see [4] or [7] for a proof.

LEMMA 4. *There is a tree of height $\aleph_1$ such that all its levels are countable and there is no path through it.* □

Finally, we are ready to produce the third of our linearly ordered sets.

LEMMA 5. *There is a linearly ordered set B of cardinality $\aleph_1$ such that*
(a) *There are no monotone sequences of length $\aleph_1$ in B.*
(b) *No uncountable subset of B has a countable pseudo-dense subset.*

PROOF. Let $(T, <)$ be a tree with height $\aleph_1$, countable levels, and no paths, as in Lemma 4. Note that then $|T|=\aleph_1$. Let $<'$ be any linear ordering of T, and let $\prec$ be as in the discussion preceding Lemma 3. We shall show that $(T, \prec)$ has both the properties required of B.

(a) Suppose $\{x_\alpha \mid \alpha<\aleph_1\}$ were a monotone sequence of length $\aleph_1$ in $(T, \prec)$. As each level of T is countable, we see that, for each $\gamma<\aleph_1$, the set $\bigcup_{h(y)=\gamma} \{z \mid z \geqq y\}$ contains uncountably many x_α's. As there are only countably many terms in this union, there must be a y, of height γ, such that uncountably many x_α are $\geqq y$.

I claim that, for each γ, there is only one such y. Suppose y' were another, and suppose, without loss of generality, that y' is left of y. Then we can successively choose ξ, η, $\zeta<\aleph_1$ such that

$x_\xi \geqq y$,

$\xi<\eta$ and $x_\eta \geqq y'$,

$\eta<\zeta$ and $x_\zeta \geqq y$.

It is easy to check that then x_η is left of both x_ξ and x_ζ, so $x_\eta \prec x_\xi$, x_ζ whereas $\xi<\eta<\zeta$. This contradicts the monotonicity of $\{x_\alpha | \alpha<\aleph_1\}$.

If y is $\leqq$ uncountably many x_α's, then so are all its predecessors. Hence, these y's form a path, contrary to the choice of T.

(b) Suppose $X\subseteq Y\subseteq T$, $|X|=\aleph_0$, $|Y|=\aleph_1$, and X is pseudo-dense in Y. The countable set $\{h(x)|x\in X\}$ is bounded above by an ordinal $\alpha<\aleph_1$. All but countably many elements of Y are in $\bigcup_{h(z)=\alpha}\{y|y\geqq z\}$, so there is a z of height α such that two distinct elements of Y are $\geqq z$. Let a and b be such elements, and let $a\prec b$. As X is pseudo-dense in Y, there is an $x\in X$ such that $a\prec x\prec b$. If $x<b$, then, as $z\leqq b$ and z has greater height than x, we find $x<z\leqq a$, so $x\prec a$, a contradiction. So x must be left of b; that is, if β is the least ordinal ($\leqq h(x)$) such that $p_\beta(x)\neq p_\beta(b)$, then $p_\beta(x)<'p_\beta(b)$. But, as $\beta\leqq h(x)<\alpha$, we see that $p_\beta(b)=p_\beta(z)=p_\beta(a)$, and β is also the least ordinal such that $p_\beta(x)\neq p_\beta(a)$. Thus, x is left of a, so $x\prec a$, a contradiction. □

Let $(A, <_A)$ and $(B, <_B)$ be as in Lemmas 2 and 5, respectively, and let $<$ be the usual ordering of $\aleph_1$. Let $f:\aleph_1\to A$ and $g:\aleph_1\to B$ be bijections, and let $h:A\to A$ be an anti-automorphism of $(A, <_A)$ by Lemma 2(c). Let $F:[\aleph_1]^2\to 4$ be as follows. If $\alpha<\beta<\aleph_1$, then

$$\begin{aligned} F\{\alpha,\beta\} &= 0 \quad \text{if } f(\alpha)<_A f(\beta) \text{ and } g(\alpha)<_B g(\beta),\\ &= 1 \quad \text{if } f(\alpha)<_A f(\beta) \text{ and } g(\alpha)>_B g(\beta),\\ &= 2 \quad \text{if } f(\alpha)>_A f(\beta) \text{ and } g(\alpha)<_B g(\beta),\\ &= 3 \quad \text{if } f(\alpha)>_A f(\beta) \text{ and } g(\alpha)>_B g(\beta). \end{aligned}$$

If $P(\aleph_1, 3)$ were true, then, by Lemma 1, there would exist $C\subseteq\aleph_1$ such that $|C|=\aleph_1$ and $F([C]^2)$ is included in a two element subset S of 4. If $S=\{0, 1\}$ or $S=\{2, 3\}$, then f maps C monotonically into A, contrary to Lemma 2(a). If $S=\{0, 2\}$ or $S=\{1, 3\}$, then g maps C monotonically into B, contrary to Lemma 5(a). If $S=\{0, 3\}$ then gf^{-1} maps the uncountable set $f(C)\subseteq A$ isomorphically to $g(C)\subseteq B$, while if $S=\{1, 2\}$, then $gf^{-1}h$ maps $h^{-1}f(C)$ isomorphically to $g(C)$. In either of the last two cases, an uncountable subset of A is isomorphic to an uncountable subset of B. A glance at Lemmas 2(b) and 5(b) shows that this is impossible. Each choice of S has led to a contradiction, so $P(\aleph_1, 3)$ cannot hold. Theorem 1 is proved. □

Theorem 1 can be generalized to apply to certain cardinals larger than $\aleph_1$. An analogue of Lemma 2, with "κ" in place of "$\aleph_1$" and "of cardinality $<\kappa$" in place of "countable", can be proved for any κ provided there exists $\mu<cf(\kappa)$ such that $\kappa\leqq 2^\mu$. The role of the real line in the proof of Lemma 2 is played by the lexicographic ordering of ${}^\mu 2$ for the least such μ, and the role of the rationals is played by the subset of ${}^\mu 2$ consisting of ultimately constant functions. A similar analogue of Lemma 5 can be proved provided κ is regular and the analogue of Lemma 4 holds. Thus, we can obtain the following result.

THEOREM 2. *If κ is regular and accessible, and if there is a tree of height κ with levels of cardinality $<\kappa$ and without paths, then not $P(\kappa, 3)$.*

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