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TWO OBSERVATIONS ON THE CONGRUENCE EXTENSION PROPERTY

G. GRÄTZER AND H. LAKSER[1]

ABSTRACT. A pair of algebras $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ is said to have the (Principal) Congruence Extension Property (abbreviated as PCEP and CEP, respectively) if every (principal) congruence relation of $\mathfrak{B}$ can be extended to $\mathfrak{A}$. A pair of algebras $\mathfrak{A}$, $\mathfrak{B}$ is constructed having PCEP but not CEP, solving a problem of A. Day. A result of A. Day states that if $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ and if for any subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ containing $\mathfrak{B}$, the pair $\mathfrak{A}$, $\mathfrak{C}$ has PCEP, then $\mathfrak{A}$, $\mathfrak{B}$ has CEP. A new proof of this theorem that avoids the use of the Axiom of Choice is also given.

1. **The example.** Let $A=\{a, b, c, d, e, f\}$. We define a binary operation $+$ on A by $a+f=e$, $b+f=e$, $x+y=x$ otherwise. Let $\mathfrak{A}=\langle A; +\rangle$ and $B=\{a, b, c, d\}$. Then $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. An easy computation shows that $\mathfrak{A}$, $\mathfrak{B}$ has PCEP. Now let $\Theta=\Theta_{\mathfrak{B}}(a, c)\vee\Theta_{\mathfrak{B}}(b, d)$. Then $c\not\equiv d(\Theta)$. However, if $\overline{\Theta}$ denotes the smallest congruence of $\mathfrak{A}$ with $\overline{\Theta}_B\geqq\Theta$, then $a\equiv c(\overline{\Theta})$; hence $a+f\equiv c+f(\overline{\Theta})$, that is, $e\equiv c(\overline{\Theta})$. Similarly, $b\equiv d(\overline{\Theta})$, and so $e\equiv d(\overline{\Theta})$. By transitivity, $c\equiv d(\overline{\Theta})$. Thus $\overline{\Theta}_B\neq\Theta$. This means that the pair $\mathfrak{A}$, $\mathfrak{B}$ does not have CEP.

2. **The proof.** We want to prove the following important[2]

THEOREM (A. DAY [**1**]). *Let $\mathfrak{A}$ be an algebra and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. If PCEP holds for any pair $\mathfrak{A}$, $\mathfrak{C}$ where $\mathfrak{C}$ contains $\mathfrak{B}$, then $\mathfrak{A}$, $\mathfrak{B}$ has CEP.*

Our proof, as well as Day's, is based on the following (A. W. Goldie [**3**], see also G. Grätzer [**4**] and Exercise 64 of Chapter 1 in G. Grätzer [**5**]):

LEMMA. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebras and let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$. Let Φ be a congruence relation of $\mathfrak{A}$ and Θ be a congruence relation of $\mathfrak{B}$ satisfying $\Phi_B\leqq\Theta$. Set $D=[B]\Phi=\{x|x\in A, x\equiv y(\Phi)$ for some $y\in B\}$. We define*

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[2] For an interesting application of this result see [2].

a binary relation $\Theta(\Phi)$ *on* D *by the rule* $u \equiv v(\Theta(\Phi))$ *iff* $u \equiv x(\Phi)$, $x \equiv y(\Theta)$, $y \equiv v(\Phi)$ *for some* $x, y \in B$. *Then* $\mathfrak{D}$ *is a subalgebra of* $\mathfrak{A}$ *and* $\Theta(\Phi)$ *is a congruence relation on* $\mathfrak{D}$. *Furthermore*, $(\Theta(\Phi))_B = \Theta$.

PROOF OF THE THEOREM. Let $\mathfrak{A}$ and $\mathfrak{B}$ be given as in the Theorem. We shall prove that, for any subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ containing $\mathfrak{B}$, the pair $\mathfrak{A}$, $\mathfrak{C}$ has CEP. Let Θ be a congruence relation on $\mathfrak{C}$ and let $\overline{\Theta}$ be the smallest congruence relation on $\mathfrak{A}$ satisfying $\overline{\Theta}_C \geqq \Theta$. Obviously,

$$\overline{\Theta} = \bigvee (\Theta_{\mathfrak{A}}(x, y) \mid x, y \in C \text{ and } x \equiv y(\Theta)).$$

We want to show that

(*) for $a, b \in C$, $a \equiv b(\overline{\Theta})$ implies that $a \equiv b(\Theta)$

(this is CEP). In view of the formula for $\overline{\Theta}$ and the way joins of congruences can be described, (*) is equivalent to:

(**) For any subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ with $B \subseteq C$, if $a, b \in C$, $a_1, b_1, \cdots, a_n, b_n \in C$, $a_i \equiv b_i(\Theta)$ for $i = 1, \cdots, n$, $a = x_0$, $x_1, \cdots, x_n = b$, $x_i \in A$ for $i = 1, \cdots, n-1$, and $x_{i-1} \equiv x_i(\Theta_{\mathfrak{A}}(a_i, b_i))$ for $i = 1, \cdots, n$, then $a \equiv b(\Theta)$.

We prove this statement by induction on n. For $n = 1$ it is obvious since $\mathfrak{A}$, $\mathfrak{C}$ has PCEP. Now assume that $n > 1$ and that the statement is valid for $n-1$. Set $D = [C]\Theta_{\mathfrak{A}}(a_n, b_n)$, $\Theta_0 = \Theta_{\mathfrak{C}}(a_1, b_1) \vee \cdots \vee \Theta_{\mathfrak{C}}(a_n, b_n)$. Since PCEP holds for $\mathfrak{A}$, $\mathfrak{C}$, we have $(\Theta_{\mathfrak{A}}(a_n, b_n))_C \leqq \Theta_0$; hence we can form $\Psi = \Theta_0(\Theta_{\mathfrak{A}}(a_n, b_n))$ and it will satisfy $\Psi_C = \Theta_0$. Now observe that $\mathfrak{A}$, $\mathfrak{D}$, $a = x_0, \cdots, x_{n-1}, a_1, b_1, \cdots, a_{n-1}, b_{n-1}$, and Ψ satisfy the assumptions of (**) with $n-1$, hence we can conclude that $a \equiv x_{n-1}(\Psi)$. Obviously, $x_{n-1} \equiv x_n(\Psi)$, hence $a \equiv b(\Psi)$. Since $a, b \in C$ and $\Psi_C = \Theta_0 \leqq \Theta$ we conclude that $a \equiv b(\Theta)$, completing the proof of (**). If we now let $\mathfrak{C} = \mathfrak{B}$ the theorem follows.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG, MANITOBA, CANADA