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TENSOR PRODUCTS OF QUATERNION ALGEBRAS*

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ABSTRACT. Two quaternion division algebras have a common quadratic subfield if their tensor product contains zero-divisors.

Let Q_1 and Q_2 be quaternion division algebras over a field K. If Q_1, Q_2 have a common quadratic subfield, it is evident that $Q_1\otimes Q_2$ contains zero-divisors (all tensor products are taken over K). In this note we shall prove that the converse is true.

THEOREM. *Let Q_1, Q_2 be quaternion division algebras with center K. Suppose that $Q_1\otimes Q_2$ is not a division algebra. Then Q_1 and Q_2 possess a common quadratic subfield.*

PROOF. Let $K(u)$ be a separable quadratic subfield of Q_2, and let the nontrivial automorphism of $K(u)$ over K be given by $u\to u'$. We complete a generation of Q_2 with an element v such that $uv=vu'$. We have that $uu'=a$ and $v^2=b$, where a and b are nonzero elements of K. Write $L=Q_1\otimes K(u)$. If L is not a division algebra, then Q_1 contains a subfield isomorphic to $K(u)$, and we are done. So the proof need only continue on the assumption that L is a division algebra.

We have that $Q_1\otimes Q_2$ is the vector space direct sum of L and Lv. The given zero-divisor thus has the form $p+qv$ with $p, q \in L$. Necessarily $q\neq 0$, and we can renormalize to make $q=1$. Write $p=c+du$, with $c, d\in Q_1$, and set $p^*=c+du'$. Note that u and v commute with c and d. Thus we have $vp=p^*v$ (and also $pv=vp^*$). Hence $(p+v)(p^*-v)=pp^*-v^2=pp^*-b$. This element is a zero-divisor and lies in L; hence it is 0. So

$$\begin{aligned} b = pp^* &= (c+du)(c+du') \\ &= c^2+d^2a+dcu+cdu' \\ &= c^2+d^2a+(dc-cd)u+cd(u+u'). \end{aligned}$$

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Since $u+u' \in K$, we deduce from this that c and d commute. We next note that

$$(pv)^2 = pvpv = pp^*v^2 = b^2.$$

Write F for the field generated over K by c and d. If $F \otimes Q_2$ is a division algebra, we can deduce from $(pv)^2 = b^2$ that $pv = \pm b$, $p = \pm v$, a contradiction. Hence $F \otimes Q_2$ is not a division algebra, and it follows that Q_2 contains a quadratic subfield isomorphic to F, as desired.

I discovered this theorem some time ago. There appears to be some continuing interest in it, and I am therefore publishing it now.

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