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QUOTIENT RINGS OF ENDOMORPHISM RINGS OF MODULES WITH ZERO SINGULAR SUBMODULE

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ABSTRACT. Throughout this paper (R, M, N, S) will denote a Morita context satisfying a certain nonsingularity condition. For such contexts we give necessary and sufficient conditions in terms of M and R for S to have a semisimple maximal left quotient ring; respectively a full linear maximal left quotient ring, a semisimple classical left quotient ring. In doing so we extend the corresponding well-known theorems for rings (employing them in the process) to endomorphism rings.

Suppose (R, M, N, S) is a Morita context ([1], [2]). That is suppose ${}_RM_S$ and ${}_SN_R$ are bimodules with an R-R bimodule homomorphism $(\, , \,): M \otimes_S N \to R$ and an S-S bimodule homomorphism $[\, , \,]: N \otimes_R M \to S$ satisfying

$$m_1[n_1, m_2] = (m_1, n_1)m_2 \quad \text{and} \quad n_1(m_1, n_2) = [n_1, m_1]n_2$$

for all $m_1, m_2 \in N$ and $n_1, n_2 \in N$.

Throughout, unless otherwise indicated, M and N will satisfy the following condition: *M_S is faithful; and $[N, m]=0$ for $m \in M$ implies that $m=0$.*

Note that when this condition is satisfied, we can (and will) assume that $S \subseteq \mathrm{Hom}_R(M, M)$.

Let ${}_RM$ be any left R-module, and set $N=\mathrm{Hom}_R(M, R)$ and $S=\mathrm{Hom}_R(M, M)$. Set $(m, f)=(m)f$ for $m \in M, f \in N$; and $[f, m]$ is defined via $m_1[f, m]=(m_1, f)m$ for all $m, m_1 \in M, f \in N$. Then (R, M, N, S) is a Morita context, called the *standard context* for ${}_RM$.

If R is semiprime and ${}_RM$ is torsionless, then the above condition is satisfied by the standard Morita context for ${}_RM$. If ${}_RM$ is a generator and $1 \in R$; or indeed, if (Trace ${}_RM)m \neq 0$ whenever $0 \neq m \in M$, then the standard Morita context for ${}_RM$ satisfies the above condition.

LEMMA 1. (a) *If A is an essential left ideal of S, then MA is an essential submodule of ${}_RM$.*

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(b) *If K is an essential submodule of $_RM$, then*

$$[N, K] = \left\{\sum_{i=1}^{t} [n_i, k_i] \mid n_i \in N, k_i \in K\right\}$$

is an essential left ideal of S.

PROOF. (a) If $0 \neq m \in M$, then $[N, m] \cap A \neq 0$. Since M_S is faithful, $0 \neq M([N, m] \cap A) \subseteq M[N, m] \cap MA = (M, N)m \cap MA \subseteq Rm \cap MA$.

(b) If $0 \neq s \in S$, then $Ms \cap K \neq 0$. Hence $0 \neq [N, Ms \cap K] \subseteq [N, Ms] \cap [N, K] \subseteq Ss \cap [N, K]$.

PROPOSITION 2. $Z(_RM)=0$ *if and only if* $Z(_SS)=0$.

PROOF. Suppose $Z(_RM)=0$, and let $s \in Z(_SS)$. Then $As=0$ where A is an essential left ideal of S. By the preceding lemma, MA is an essential submodule of $_RM$, and clearly $(MA)s=0$. Hence ker s is essential in $_RM$, and since $Z(_RM)=0$ it follows that $s=0$.

Conversely suppose that $Z(_RM) \neq 0$. If $0 \neq z \in Z(_RM)$ it follows from the condition on [,] that there is an $n \in N$ with $0 \neq [n, z]$. Set $s=[n, z]$. We claim that $s \in Z(_SS)$, and by the previous lemma it suffices to prove that ker s is essential in $_RM$. If $m \in M$, $m \notin$ ker s, then $(m, n) \neq 0$. Since $z \in Z(_RM)$ there exists $a \in R^1$ (R^1 denotes R with identity adjoined in the customary manner) with $a(m, n) \neq 0$ and $a(m, n)z=0$. But then $ams = am[n, z]=a(m, n)z=0$. Hence $0 \neq am \in$ ker s and ker s is essential in $_RM$.

PROPOSITION 3 [**1**, p. 276]. $d(_RM)=d(_SS)$.

PROOF. Here $d(_RM)$ denotes the (Goldie) dimension of $_RM$. If $\bigoplus_i A_i$ is an internal direct sum of nonzero left ideals of S, then a routine calculation shows that $\sum_i MA_i$ is a direct sum of nonzero submodules of $_RM$. Hence $d(_SS) \leqq d(_RM)$. On the other hand if $\bigoplus_i M_i$ is an internal direct sum of nonzero submodules of $_RM$ and $A_i=[N, M_i]$, then $\sum_i A_i$ is a direct sum of nonzero left ideals of S. Hence $d(_RM) \leqq d(_SS)$.

Let $\Lambda=\mathrm{Hom}_R(M, M)$ and $\Omega=\mathrm{Hom}_R(\bar{M}, \bar{M})$ where $\bar{M}$ is the injective hull of $_RM$. As we have already noted, we can assume that $S \subseteq \Lambda$. When $Z(_RM)=0$, Ω is a regular self-injective ring [**5**] and we can assume that $\Lambda \subseteq \Omega$.

PROPOSITION 4. *When $Z(_RM)=0$, Ω is the maximal left quotient ring of S.*

PROOF. Given $0 \neq \omega \in \Omega$, $M\omega^{-1} \cap M$ is an essential submodule of $_RM$, and so $(M\omega^{-1} \cap M)\omega \neq 0$. Hence

$$0 \neq [N, (M\omega^{-1} \cap M)\omega] = [N, M\omega^{-1} \cap M]\omega \subseteq S \cap S\omega.$$

Since $Z({}_S S)=0$ it follows that Ω is a maximal left quotient ring of S.

THEOREM 5. *S has a semisimple maximal left quotient ring (necessarily isomorphic to Ω) if and only if $Z({}_R M)=0$ and $d({}_R M)<\infty$.*

PROOF. By [**8**, Theorem 1.6], S has a semisimple maximal left quotient ring if and only if $Z({}_S S)=0$ and $d({}_S S)<\infty$. The theorem then follows from the previous three propositions.

A submodule K of ${}_R M$ is *closed* if K has no proper essential extensions in ${}_R M$. Let $\mathscr{C}({}_R M)$ denote the set of closed submodules of ${}_R M$. If $Z({}_R M)=0$ it is well known that $\mathscr{C}({}_R M)$ is a complete complemented lattice and $\mathscr{C}({}_R M)$ is lattice isomorphic to $\mathscr{C}({}_R \bar{M})$ under contraction [**3**, p. 61].

PROPOSITION 6. *For any module ${}_R M$ (not necessarily satisfying the standing hypothesis) with $Z({}_R M)=0$:*

(a) *If $A\in\mathscr{C}({}_\Omega\Omega)$, then $\bar{M}A\in\mathscr{C}({}_R\bar{M})$.*

(b) *If $K\in\mathscr{C}({}_R\bar{M})$, then $\mathrm{Hom}_R(\bar{M}, K)\in\mathscr{C}({}_\Omega\Omega)$.*

(c) *$\mathscr{C}({}_R\bar{M})$ is lattice isomorphic to $\mathscr{C}({}_\Omega\Omega)$.*

PROOF. (a) Since Ω is regular left self-injective $A=\Omega\varepsilon$ where $\varepsilon^2=\varepsilon\in\Omega$. Then $\bar{M}A=\bar{M}\varepsilon$, a direct summand of ${}_R\bar{M}$ and hence closed.

(b) $K=\bar{M}\varepsilon$ for some $\varepsilon^2=\varepsilon\in\Omega$, and then $\mathrm{Hom}_R(\bar{M}, K)=\Omega\varepsilon$ which is closed.

(c) This follows from the preceding correspondence.

COROLLARY 7. *If $Z({}_R M)=0$, then $\mathscr{C}({}_R M)$ is lattice isomorphic to $\mathscr{C}({}_S S)$.*

PROOF. $\mathscr{C}({}_S S)\cong\mathscr{C}({}_S\Omega)\cong\mathscr{C}({}_\Omega\Omega)$ by [**3**, p. 61 and p. 70] and Proposition 4. Hence, by Proposition 6, $\mathscr{C}({}_R M)\cong\mathscr{C}({}_R\bar{M})\cong\mathscr{C}({}_\Omega\Omega)\cong\mathscr{C}({}_S S)$.

A module ${}_R M$ is *atomic* if each nonzero element of $\mathscr{C}({}_R M)$ contains a minimal nonzero element of $\mathscr{C}({}_R M)$, called an *atom*. A ring is a *full linear ring* if it is isomorphic to the full ring of linear transformations of a left vector space over a division ring.

THEOREM 8. *S has a maximal left quotient ring which is a direct product of full linear rings if and only if $Z({}_R M)=0$ and ${}_R M$ is atomic.*

PROOF. By [**6**, Theorem 2], S has a maximal left quotient ring which is a direct product of full linear rings if and only if $Z({}_S S)=0$ and ${}_S S$ is atomic. By virtue of Corollary 7, ${}_S S$ is atomic if and only if ${}_R M$ is. The result follows from Proposition 2.

A module ${}_R M$ is *Q-prime* if for any atoms K_1 and K_2 of $\mathscr{C}({}_R M)$ there exist nonzero isomorphic submodules of K_1 and K_2 respectively.

PROPOSITION 9. *Suppose $Z({}_R M)=0$ and ${}_R M$ is atomic. Then ${}_R M$ is Q-prime if and only if all atoms of $\mathscr{C}({}_R\bar{M})$ are isomorphic; equivalently* *if and only if all atoms of* $\mathscr{C}({}_\Omega\Omega)$ *are isomorphic.* (*For the preceding, the standing hypothesis need not hold.*) *Consequently,* ${}_RM$ *is Q-prime if and only if* ${}_SS$ *is Q-prime.*

PROOF. If ${}_RM$ is Q-prime and K_1 and K_2 are atoms of ${}_R\bar{M}$, then $K_1 \cap M$ and $K_2 \cap M$ are atoms of ${}_RM$. $K_1 \cap M$ and $K_2 \cap M$ contain nonzero isomorphic submodules, and so $K_1 \cong K_2$ since K_1 and K_2 are injective. Conversely, suppose all atoms of ${}_R\bar{M}$ are isomorphic. If L_1 and L_2 are atoms of ${}_RM$, then there exist isomorphic atoms K_1 and K_2 of ${}_R\bar{M}$ such that $K_i \cap M = L_i$, $i=1, 2$. If $f: K_1 \to K_2$ is an isomorphism, then $X_1 = L_2 f^{-1} \cap L_1$ and $X_2 = L_1 f \cap L_2$ are nonzero isomorphic submodules of L_1 and L_2. Hence ${}_RM$ is Q-prime.

Now suppose that all atoms of ${}_R\bar{M}$ are isomorphic and A_1, A_2 are atoms of $\mathscr{C}({}_\Omega\Omega)$. By Proposition 6, $\bar{M}A_1$ and $\bar{M}A_2$ are in $\mathscr{C}({}_R\bar{M})$. Let $\varphi: \bar{M}A_1 \to \bar{M}A_2$ be an isomorphism. Define θ from $A_1 = \mathrm{Hom}_R(\bar{M}, \bar{M}A_1)$ into $A_2 = \mathrm{Hom}_R(\bar{M}, \bar{M}A_2)$ by $\psi\theta = \psi \cdot \varphi$ for $\psi \in A_1$. A routine calculation shows that θ is an isomorphism from A_1 onto A_2.

Suppose all atoms of ${}_\Omega\Omega$ are isomorphic and let K_1, K_2 be atoms of $\mathscr{C}({}_R\bar{M})$. As in Proposition 6, $K_i = \bar{M}\varepsilon_i$ where

$$\varepsilon_i^2 = \varepsilon_i \in \Omega \quad \text{and} \quad \mathrm{Hom}_R(\bar{M}, K_i) = \Omega\varepsilon_i \in \mathscr{C}({}_\Omega\Omega)$$

for $i=1, 2$. Since $\Omega\varepsilon_1 \cong \Omega\varepsilon_2$ there exist μ, $\nu \in \Omega$ such that $\mu\nu = \varepsilon_1$ and $\nu\mu = \varepsilon_2$ [7, p. 63]. If $\mu' = \mu|_{K_1}$ and $\nu' = \nu|_{K_2}$, then $\mu'\nu' = \mathrm{id}_{K_1}$ and $\nu'\mu' = \mathrm{id}_{K_2}$. Thus $K_1 \cong K_2$ and all atoms of ${}_R\bar{M}$ are isomorphic.

THEOREM 10. *S has a maximal left quotient ring which is a full linear ring if and only if* $Z({}_RM)=0$, ${}_RM$ *is atomic, and* ${}_RM$ *is Q-prime.*

PROOF. By previous propositions ${}_RM$ has the above properties if and only if ${}_SS$ does. By [**6**, Theorem 1] this is the case exactly when S has a full linear ring as its maximal left quotient ring.

THEOREM 11. *S has a classical left quotient ring which is simple* (*semisimple*) *with minimum condition if and only if* $\bar{R} = R/\mathrm{ann}\,{}_RM$ *is prime* (*semiprime*), $m[N, M]=0$ *for* $m \in M$ *implies that* $m=0$, $Z({}_RM)=0$, *and* $d({}_RM) < \infty$.

PROOF. By a well-known theorem [**4**], S has a classical left quotient ring which is simple (semisimple) with minimum condition if and only if S is a prime (semiprime) ring, $Z({}_SS)=0$, and $d({}_SS) < \infty$. In view of the earlier propositions, it suffices to prove that S is prime (semiprime) exactly when $\bar{R}$ is prime (semiprime) and $m[N, M]=0$ for $m \in M$ implies that $m=0$.

Suppose that S is prime and let r, $r' \in R \setminus \mathrm{ann}\,{}_RM$. Then $rM \neq 0$, $r'M \neq 0$, and consequently $[N, rM] \neq 0$, $[N, r'M] \neq 0$. Since S is prime

$0\neq[N, rM][N, r'M]=[N, r(M, N)r'M]$. In particular $r(M, N)r'M\neq 0$ proving that $\bar{r}(\overline{M, N})\bar{r}'\neq 0$ in $\bar{R}$ ($\bar{r}$ denotes the coset of r in $\bar{R}$). Hence $\bar{R}$ is prime. Next suppose that $m[N, M]=0$ with $m\in M$. Then $[N, m][N, M]=0$, and since S is prime $[N, m]=0$ proving that $m=0$.

Conversely, suppose that $\bar{R}$ is prime and that $m[N, M]=0$ for $m\in M$ implies that $m=0$. Let $0\neq s\in S$, $0\neq t\in S$. Then $Ms\neq 0$ and $Mt\neq 0$ so $Ms[N, M]\neq 0$ and $Mt[N, M]\neq 0$. Thus $(Ms, N)\nsubseteq \operatorname{ann}_R M$ and $(Mt, N)\nsubseteq \operatorname{ann}_R M$. Since $\bar{R}$ is prime $(Ms, N)(Mt, N)=M(s[N, M]t, N)\nsubseteq \operatorname{ann}_R M$. In particular, $s[N, M]t\neq 0$ proving that S is prime.

The semiprime case is obtained by taking $r=r'$ and $s=t$ in the above proof.

We remark that [**9**, Theorem 2.3] is a special case of the preceding theorem.

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