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ON AN INDUCTION THEOREM FOR RELATIVE GROTHENDIECK GROUPS[1]

WILLIAM H. GUSTAFSON

ABSTRACT. We present an improvement in the proof of Dress' induction theorem for relative Grothendieck rings.

1. **Introduction.** Let G be a finite group, $\mathfrak{U}$ a collection of subgroups of G and R a commutative ring with identity element. Let $\mathfrak{M}$ denote the category of finitely generated (left) modules over the group ring RG. For $M \in \text{obj}(\mathfrak{M})$, let $[M]$ denote the isomorphism class of M. Let A denote the free abelian group generated by all $[M]$, $M \in \text{obj}(\mathfrak{M})$, and let B denote the subgroup of A generated by all $[M]-[M']-[M'']$ such that there is an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

in $\mathfrak{M}$ which splits upon restriction to RH for each $H \in \mathfrak{U}$. The quotient group A/B is the *relative Grothendieck group of RG relative to* $\mathfrak{U}$, denoted by $a_R(G, \mathfrak{U})$.

Dress **[2]** has given some induction theorems for $a_R(G, \mathfrak{U})$, in the spirit of Artin's induction theorem for rational characters of G. Dress' results depend on a crucial proposition which is proved with some difficulty. In this note, we show that the proposition follows readily from **[4]**, provided that the ground ring R is an algebra (not necessarily faithful) over the ring of p-adic integers. Thus our proof is applicable to the important modular case.

2. **The main proposition.** By a *G-set*, we mean a finite set on which G acts from the left by permutations. Every G-set can be written uniquely as a disjoint union of sub-G-sets on which G acts transitively. Further, if S is a transitive G-set, then S is isomorphic (in the category of G-sets and G-equivariant set maps) to a G-set of the form $G/\!\!/H$, i.e., the collection of left cosets of a subgroup H of G, with the natural action of G.

For any G-set S, we may define a finitely generated left RG-module RS as follows: RS is the free R-module with S as basis, and with action of G determined by the G-set structure of S. In particular, if $S=G$ with

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action given by left multiplication, then $RS \simeq RG$ as left RG-modules. If S consists of one element, with each element of G acting as the identity, then RS is just the trivial representation of G over R. The proposition of Dress to which we have alluded above is

PROPOSITION 1 (DRESS [2, p. 93]). *Let R be any commutative ring with unit, let G be an elementary abelian p-group of type (p, p) and let H be a subgroup of G of order p. Then the following relation holds in $a_R(G, H)$ $(=a_R(G, \{H\}))$:*

$$p \cdot [R] + [RG] = \sum_V [R(G/\!\!/V)],$$

where the sum on the right extends over all the subgroups V of order p in G.

Dress' proof requires a detailed and delicate analysis of the structure of RG- and RH-modules.

3. **Proposition 1 when R is a p-adic algebra.** In this section we prove Proposition 1 in the case where R is an algebra over the valuation ring Z_p^* in the p-adic completion of the rationals. It is clear that we may assume that $R=Z_p^*$, and that $a_R(G, H)$ is calculated from the category of RG-lattices (i.e. finitely generated left RG-modules which are free as R-modules). This calculation has been given in detail in [4]; let us recall what was found there. We present G as $\langle a, b | a^p=b^p=[a, b]=1\rangle$. Without loss of generality, we may take $H=\langle a\rangle$. Denote by C the companion matrix of the cyclotomic polynomial $\Phi_p(X)$. Thus

$$C = \begin{pmatrix} 0 & 0 & \cdot & \cdot & 0 & -1 \\ 1 & 0 & \cdot & \cdot & 0 & -1 \\ 0 & 1 & \cdot & \cdot & 0 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 & -1 \end{pmatrix},$$

a $(p-1)\times(p-1)$ matrix. The R-irreducible RG-lattices are the trivial lattice $T\simeq R$ and lattices $W, X_0, \cdots, X_{p-1}$ which afford matrix representations as follows:

$$W: a \to I_{(p-1)\times(p-1)}, \qquad b \to C,$$
$$X_k: a \to C, \qquad b \to C^k \ (k = 0, \cdots, p-1).$$

Then $a_R(G, H)$ is a free abelian group with basis $[T]$, $[W]$, $[X_0], \cdots,$ $[X_{p-1}]$, $[(X_0, T)]$, where (X_0, T) is a certain extension of X_0 by T described in detail in [4]. We define certain group homomorphisms as follows:

$$\pi : a_R(G, H) \to a_R(G, \{1\}),\ \pi[M] = [M],$$
$$\lambda : a_R(G, \{1\}) \to a_R(G, H),\ \lambda[I] = [I] \quad \text{for } I\ R\text{-irreducible},$$
$$\text{res} : a_R(G, H) \to a_R(H, H),\ \text{res}[M] = [M_H],$$
$$\varphi : a_R(G, H) \to a_R(G, 1) \oplus a_R(H, H),\ \varphi[M] = (\pi[M], \text{res}[M]).$$

For any RG-lattice M, define $t_M \in a_R(G, H)$ by

$$t_M = \lambda\pi([M]) + \nu \cdot ([X_0, T] - [X_0] - [T]),$$

where ν is determined by

$$\text{(1)} \qquad \text{res}[M] = \nu \cdot [RH] + a \cdot [A] + b \cdot [B],$$

$A = R$ and $B = R[1^{1/p}]$ being the irreducible RH-lattices.

PROPOSITION 2. *For any RG-lattice M, $t_M = [M]$ in $a_R(G, H)$.*

PROOF. Assume res$[M]$ is given by equation (1) above. By the proof of Theorem 4.7 of [4], φ is a monomorphism. Hence it suffices to show that $\pi[M] = \pi(t_M)$ and res$[M] =$ res t_M.

Suppose that $\pi[M] = t[T] + \sum x_i[X_i] + w \cdot [W]$. Then $\lambda\pi[M]$ is given by the same formula, but lies in $a_R(G, H)$. Recalling that $T_H \cong A$, $X_i|_H \cong B$ and $W_H \cong A \oplus \cdots \oplus A$ ($p-1$ copies), we see that

$$M_H \cong \nu \cdot RH + \left(-\nu + \sum x_i\right)B + (t - \nu + (p-1)w)A.$$

By (1) and the Krull-Schmidt theorem for RH-lattices (see [1, Theorem 76.26]), we have

$$b = -\nu + \sum x_i, \qquad a = t - \nu + (p-1)w.$$

We also have

$$\pi(t_M) = \pi\lambda\pi([M]) + \nu \cdot \pi([(X_0, T)] - [X_0] - [T]) = \pi([M]),$$

since clearly $\pi([(X_0, T)] - [X_0] - [T]) = 0$.

On the other hand,

$$\begin{aligned} \text{res } t_M &= \text{res } \lambda\pi[M] + \nu \cdot \text{res}([(X_0, T)] - [X_0] - [T]) \\ &= \text{res}\left(t[T] + \sum x_i[X_i] + w[W]\right) + \nu \cdot ([RH] - [B] - [A]) \\ &= (t - \nu + (p-1)w)[A] + \left(-\nu + \sum x_i\right)[B] + \nu \cdot [RH] \\ &= a[A] + b[B] + \nu[RH] = \text{res}[M], \end{aligned}$$

whence the proposition is established.

COROLLARY. *In $a_R(G, H)$, we have*

$$[RG] + p[R] = [T] + (1 - p)[X_0] + \sum_{i \neq 0} [X_i] + [W] + p \cdot [(X_0, T)].$$

Now we will use Proposition 2 to calculate the right-hand side of the equation in Proposition 1. Let $H_k = \langle a^k b \rangle$, for $k = 1, \cdots, p$. Then a full set of subgroups of order p is $H, H_1, \cdots, H_p$. The cosets of H are H, bH, $b^2H, \cdots, b^{p-1}H$, from which we see easily that $R(G/\!\!/H)$ affords the matrix representation

$$a \to I_{p \times p}, \qquad b \to \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 \end{pmatrix}.$$

By a change of basis, this representation may be brought to the form

$$a \to I_{p \times p}, \qquad b \to \left(\begin{array}{c|c} 1 & * \\ \hline 0 & C \end{array}\right).$$

Hence we have

$$\operatorname{res}[R(G/\!\!/H)] = p[A], \qquad \lambda\pi[R(G/\!\!/H)] = [T] + [W].$$

The cosets of H_k are $H_k, aH_k, \cdots, a^{p-1}H_k$, and one has $a^i b^j \in a^{i-kj} H_k$. From this it follows that $R(G/\!\!/H_k)$ affords the matrix representation $a \to \Gamma$, $b \to \Gamma^{p-k}$ where

$$\Gamma = \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 \end{pmatrix}_{p \times p}.$$

By change of basis, Γ may be transformed to

$$\left(\begin{array}{c|c} 1 & * \\ \hline 0 & C \end{array}\right),$$

and the same basis change transforms Γ^{p-k} to

$$\left(\begin{array}{c|c} 1 & *' \\ \hline 0 & C^{p-k} \end{array}\right).$$

Thus we see that

$$\operatorname{res}[R(G/\!\!/H_k)] = [RH], \qquad \lambda\pi[R(G/\!\!/H_k)] = [T] + [X_{p-k}].$$

Thus it follows from Proposition 2 that

$$\begin{aligned}\sum_{|V|=p} [R(G/\!\!/V)] &= [T] + [W] + p \cdot ([(X_0, T)] - [X_0] - [T]) \\ &\quad + p[T] + \sum_{i=0}^{p-1} [X_i] \\ &= [T] + [W] + p[(X_0, T)] + (1-p)[X_0] \\ &\quad + \sum_{i \neq 0} [X_i].\end{aligned}$$

We may now deduce Proposition 1 by comparing equation (2) and the Corollary to Proposition 2.

4. **Descending to the integers.** It would be pleasing to use this method to obtain the full force of Proposition 1. In order to do so, it would suffice to handle the case in which R is the ring Z of rational integers. Dress [**3**, Satz 1] has shown that, for the localization Z_p of Z at p, the map $a_{Z_p}(G, \mathfrak{U}) \to a_{Z_p^*}(G, \mathfrak{U})$ induced by the completion functor is monic. Hence our proof transfers to $a_{Z_p}(G, \mathfrak{U})$. (Indeed, most of the results of [**4**] can now be taken with coefficients from Z_p.) Dress has further indicated to me in a private communication that he has a method of descent from Z_p to Z. This method will be discussed in his forthcoming lecture notes from Universität Bielefeld.

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Department of Mathematics, Indiana University, Bloomington, Indiana 47401

Current address: Department of Mathematics, Brandeis University, Waltham, Massachusetts 02154