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FUNCTIONS WITH REAL PART GREATER THAN α

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ABSTRACT. Let $\mathscr{P}_b(\alpha)$ denote the class of functions $P(z)=1+b(1-\alpha)z+\cdots$ which are analytic and satisfy $\operatorname{Re}\{P(z)\}>\alpha$ for $|z|<1$ where $\alpha\in[0,1)$ and $b\in[0,2]$. We demonstrate some inequalities involving $|P'(z)|$ and $|P'(z)/P(z)|$ dependent on b and α which are subsequently applied to the class of functions whose derivative lies in $\mathscr{P}_b(\alpha)$ to obtain distortion, covering, and radius of convexity properties.

1. **Introduction.** Let $\mathscr{P}(\alpha)$ denote the class of functions $P(z)=1+b_1z+\cdots$ which are analytic and satisfy $\operatorname{Re}\{P(z)\}>\alpha$ for $|z|<1$ where $\alpha\in[0,1)$. If $\varepsilon=\exp\{-\arg b_1\}$, then $P(\varepsilon z)=1+|b_1|z+\cdots$ and we see that to limit our study of $\mathscr{P}(\alpha)$ to functions with a nonnegative real first coefficient is actually no restriction. Also, it is known [**5**, p. 170] that $|b_1|\leqq 2(1-\alpha)$ so as a notational and computational device we define $\mathscr{P}_b(\alpha)=\{P(z)\in\mathscr{P}(\alpha):P'(0)=b(1-\alpha)\}$ for $b\in[0,2]$. In this paper inequalities involving $\operatorname{Re}\{P(z)\}$, $|P(z)|$, and $|P'(z)/P(z)|$ dependent on b are obtained and applied to functions whose derivative has real part greater than α. Results include distortion, covering, and convexity properties which extend some theorems of MacGregor [**4**] who considered the $\alpha=0$ case.

2. **The class $\mathscr{P}_b(\alpha)$.** Let $\mathscr{A}$ denote the class of functions $\phi(z)$ which are analytic and satisfy $|\phi(z)|\leqq 1$ for $|z|<1$. The following lemmas will be needed in the proofs of our theorems. The first may be found in Nehari [**5**, p. 167] and the second is due to Tepper [**6**, p. 520].

LEMMA 1. *If $\phi(z)\in\mathscr{A}$ and $\phi(0)=z_0$, then*

(i) $$|\phi'(z)|\leqq(1-|\phi(z)|^2)/(1-|z|^2),$$

(ii) $$(|z_0|-|z|)/(1-|z_0z|)\leqq|\phi(z)|\leqq(|z_0|+|z|)/(1+|z_0z|).$$

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LEMMA 2. *If* $P(z) \in \mathscr{P}_b(0)$, *then*

(i) $$|P(z)| \leqq (1 + b\,|z| + |z|^2)/(1 - |z|^2),$$

(ii) $$\operatorname{Re}\{P(z)\} \geqq (1 - |z|^2)/(1 + b\,|z| + |z|^2).$$

We will extend Lemma 2 to the α case as an immediate corollary which will be utilized in the next section.

COROLLARY 1. *If* $P(z) \in \mathscr{P}_b(\alpha)$, *then*

(i) $$|P(z)| \leqq (1 + b(1 - \alpha)\,|z| + (1 - 2\alpha)\,|z|^2)/(1 - |z|^2),$$

(ii) $$\operatorname{Re}\{P(z)\} \geqq (1 + b\alpha\,|z| - (1 - 2\alpha)\,|z|^2)/(1 + b\,|z| + |z|^2).$$

The result is sharp.

PROOF. Let $H(z)=(P(z)-\alpha)/(1-\alpha)$, then $H(z) \in \mathscr{P}_b(0)$ and $P(z)=(1-\alpha)H(z)+\alpha$. Apply Lemma 2 (i) to $|P(z)|\leqq(1-\alpha)|H(z)|+\alpha$ and Lemma 2(ii) to $\operatorname{Re}\{P(z)\}=(1-\alpha)\operatorname{Re}\{H(z)\}+\alpha$ and simplify. Furthermore, the result is sharp for all $b \in [0, 2]$ by considering the functions $P_1(z)=(1+b(1-\alpha)z+(1-2\alpha)z^2)(1-z^2)^{-1}$ for (i) and $P_2(z)=(1-b\alpha z-(1-2\alpha)z^2)\cdot(1-bz+z^2)^{-1}$ for (ii).

THEOREM 1. *If* $P(z) \in \mathscr{P}_b(0)$, *then*

$$|P'(z)| \leqq \frac{\operatorname{Re}\{P(z)\}}{1-|z|^2}\left\{\frac{b\,|z|^2 + 4\,|z| + b}{|z|^2 + b\,|z| + 1}\right\}.$$

PROOF. Since $\operatorname{Re}\{P(z)\}>0$ and $P(0)=1$ the function $P(z)$ is subordinate to the function $p(z)=(1+z)/(1-z)$; therefore there exists a function $w(z) \in \mathscr{A}$ such that

(1) $$P(z) = (1 + w(z))/(1 - w(z)) = 1 + bz + \cdots.$$

Direct computation shows that $w(z)=bz/2+\cdots$ and furthermore, since $w(0)=0$ and $|w(z)| \leqq 1$, there exists $\phi(z) \in A$ such that $w(z)=z\phi(z)$. After substituting, the quotient in (1) becomes

(2) $$P(z) = (1 + z\phi(z))/(1 - z\phi(z)).$$

Taking the derivative of both sides of (2) with respect to z and then taking the absolute value of both sides we get

(3) $$|P'(z)| = \frac{2\,|z\phi'(z) + \phi(z)|}{|1 - z\phi(z)|^2} = \frac{2\,|z\phi'(z) + \phi(z)|}{1 - |z\phi(z)|^2}\,\frac{1 - |z\phi(z)|^2}{|1 - z\phi(z)|^2}$$
$$= \frac{2\,|z\phi'(z) + \phi(z)|}{1 - |z\phi(z)|^2}\operatorname{Re}\{P(z)\}.$$

An application of the triangle inequality and then Lemma 1 (i) to the RHS of (3) yields

$$(4)\quad \begin{aligned} |P'(z)| &\leqq \frac{2\,\mathrm{Re}\{P(z)\}}{1-|z|^2}\left\{\frac{|\phi(z)|\,(1-|z|^2)+|z|\,(1-|\phi(z)|^2)}{1-|z|^2\,|\phi(z)|^2}\right\} \\ &= \frac{2\,\mathrm{Re}\{P(z)\}}{1-|z|^2}\left\{\frac{|\phi(z)|+|z|}{1+|z|\,|\phi(z)|}\right\}. \end{aligned}$$

We claim that the expression in braces on the RHS of (4) is monotone increasing with respect to $|\phi(z)|$. Let $r=|z|$, $x=|\phi(z)|$, and $g(x)=(x+r)/(1+xr)$; then $g'(x)=(1-r^2)/(1+xr)^2$ so $g'(x)>0$ because $r\in[0,1)$, $x\in[0,1]$. Lastly, apply Lemma 1 (ii) where $\phi(z)=w(z)/z=b/2+\cdots$ and $z_0=b/2$ to the RHS of (4):

$$\begin{aligned} |P'(z)| &\leqq \frac{2\,\mathrm{Re}\{P(z)\}}{1-|z|^2}\left\{\frac{(2\,|z|+b)/(2+b\,|z|)+|z|}{1+|z|\,(2\,|z|+b)/(2+b\,|z|)}\right\} \\ &= \frac{\mathrm{Re}\{P(z)\}}{1-|z|^2}\left\{\frac{b\,|z|^2+4\,|z|+b}{|z|^2+b\,|z|+1}\right\} \end{aligned}$$

and the proof is complete.

Corollary 2. *If $P(z)\in\mathscr{P}_b(\alpha)$, then*

$$(5)\quad |P'(z)| \leqq \frac{\mathrm{Re}\{P(z)\}-\alpha}{1-|z|^2}\left\{\frac{b\,|z|^2+4\,|z|+b}{|z|^2+b\,|z|+1}\right\}$$

Proof. Let $H(z)=(P(z)-\alpha)/(1-\alpha)$, then $H(z)\in\mathscr{P}_b(0)$ and we may use Theorem 1 to obtain

$$\frac{|P'(z)|}{1-\alpha} \leqq \mathrm{Re}\left\{\frac{P(z)-\alpha}{1-\alpha}\right\}\frac{1}{1-|z|^2}\left\{\frac{b\,|z|^2+4\,|z|+b}{|z|^2+b\,|z|+1}\right\}$$

which is equivalent to (5).

The next two corollaries become obvious by noting that $b\leqq 2$ implies that $(b|z|^2+4|z|+b)/(|z|^2+b|z|+1)\leqq 2$.

Corollary 3. *If $P(z)\in\mathscr{P}(\alpha)$, then $|P'(z)|\leqq 2(\mathrm{Re}\{P(z)\}-\alpha)/(1-|z|^2)$.*

Corollary 4. *If $P(z)\in\mathscr{P}(\alpha)$ and $P'(0)=0$, then $|P'(z)|\leqq 4|z|/(1-|z|^4)\cdot(\mathrm{Re}\{P(z)\}-\alpha)$.*

Theorem 2. *If $P(z)\in\mathscr{P}_b(\alpha)$, then*

$$\left|\frac{P'(z)}{P(z)}\right| \leqq \frac{(1-\alpha)}{1-|z|^2}\left\{\frac{b\,|z|^2+4\,|z|+b}{(1-2\alpha)\,|z|^2+b(1-\alpha)\,|z|+1}\right\}.$$

PROOF. Let $H(z)=(P(z)-\alpha)/(1-\alpha)$, then $H(z)\in\mathscr{P}_b(0)$ and $P(z)=(1-\alpha)H(z)+\alpha$. So

$$\left|\frac{P'(z)}{P(z)}\right|=\left|\frac{(1-\alpha)H'(z)}{(1-\alpha)H(z)+\alpha}\right|=\left|\frac{H'(z)}{H(z)+\alpha/(1-\alpha)}\right|. \tag{6}$$

Apply Theorem 1 to the numerator of the RHS of (6) to get

$$\begin{aligned}\left|\frac{H'(z)}{H(z)+\alpha/(1-\alpha)}\right| &\leqq \frac{1}{1-|z|^2}\left\{\frac{b\,|z|^2+4\,|z|+b}{|z|^2+b\,|z|+1}\right\}\frac{\operatorname{Re}\{H(z)\}}{|H(z)+\alpha/(1-\alpha)|}\\ &\leqq \frac{1}{1-|z|^2}\left\{\frac{b\,|z|^2+4\,|z|+b}{|z|^2+b\,|z|+1}\right\}\frac{1}{1+\dfrac{\alpha/(1-\alpha)}{\operatorname{Re}\{H(z)\}}}.\end{aligned}$$

Since $\operatorname{Re}\{H(z)\}\leqq|H(z)|$ we may use Lemma 2 (i) on $\operatorname{Re}\{H(z)\}$ in the last inequality. A simplification proves the theorem. That the result is sharp can be seen by considering the function

$$P(z)=(1+b(1-\alpha)z+(1-2\alpha)z^2)/(1-z^2).$$

3. **The class** $\mathscr{P}'_a(\alpha)$. We formally define $\mathscr{P}'_a(\alpha)$ to be the class of functions $F(z)=z+a(1-\alpha)z^2+\cdots$ such that $F'(z)\in\mathscr{P}_{2a}(\alpha)$. The class has previously been referred to as early as 1915 by Alexander **[1]** and more recently by MacGregor **[4]**. Both considered the case $\alpha=0$ without preassigned second coefficient. The results which follow are more analogous to those of the latter paper. Our first results concern distortion theorems and regions covered by functions in the class $\mathscr{P}'_a(\alpha)$.

THEOREM 3. *If* $F(z)\in\mathscr{P}'_a(\alpha)$, *then*

(i) $$|F'(z)|\leqq(1+2a(1-\alpha)\,|z|+(1-2\alpha)\,|z|^2)/(1-|z|^2),$$

(ii) $$\operatorname{Re}\{F'(z)\}\geqq(1+2a\alpha\,|z|-(1-2\alpha)\,|z|^2)/(1+2a\,|z|+|z|^2),$$

(iii) $$\begin{aligned}|F(z)|\leqq&-(1-2\alpha)\,|z|\\ &+(1-\alpha)[(1-a)\cdot\log(1+|z|)-(1+a)\cdot\log(1-|z|)],\end{aligned}$$

(iv) $$\begin{aligned}|F(z)|&\geqq-(1-2\alpha)\,|z|+a(1-\alpha)\cdot\log(|z|^2+2a\,|z|+1)\\ &\geqq+2(1-\alpha)(1-a^2)^{1/2}\{\arctan(|z|+a)/(1-a^2)^{1/2}\\ &\qquad-\arctan(a/(1-a^2)^{1/2})\}, \quad \textit{if } a\neq 1,\\ &\geqq-(1-2\alpha)\,|z|+2(1-\alpha)\cdot\log(1+|z|), \quad \textit{if } a=1.\end{aligned}$$

The result is sharp.

PROOF. (i) and (ii) follow as immediate consequences of Corollary 1. The other two follow from the fact that

$$F(z) = \int_0^z F'(\zeta)\, d\zeta = \int_0^{|z|} F'(te^{i\theta})e^{i\theta}\, dt.$$

(i) and (ii) are sharp by Corollary 1. For (iii) consider the function $F(z) = -(1-2\alpha)z+(1-\alpha)[(1-a)\cdot\log(1+z)-(1+a)\cdot\log(1-z)]$. Lastly, (iv) is sharp for

$$\begin{aligned} F_1(z) = {} & -(1-2\alpha)z + a(1-\alpha)\cdot\log(z^2+2az+1) \\ & + 2(1-\alpha)(1-a^2)^{1/2}\{\arctan(z+a)/(1-a^2)^{1/2} \\ & - \arctan(a/(1-a^2)^{1/2})\} \end{aligned}$$

if $a \neq 1$, and $F_1(z) = -(1-2\alpha)z+2(1-\alpha)\cdot\log(1+z)$ if $a=1$.

COROLLARY 5. *Each function in* $\mathscr{P}'_a(\alpha)$ *maps* $|z|<1$ *onto a domain which covers the disk*

$$\begin{aligned} |w| < {} & a(1-\alpha)\cdot\log(2+2a) + 2(1-\alpha)(1-a^2)^{1/2} \\ & \{\arctan(1+a)/(1-a^2)^{1/2} - \arctan(a/(1-a^2)^{1/2})\} - (1-2\alpha), \\ & \textit{if } a \neq 1, \end{aligned}$$

and

$$|w| < 2(1-\alpha)\cdot\log(2) - (1-2\alpha), \quad \textit{if } a = 1.$$

The result is sharp.

PROOF. In Theorem 3 let $|z|\to 1$ and the right-hand side of (iv) approach the desired quantity. For sharpness consider the function $F_1(z)$ in the proof of Theorem 3.

THEOREM 4. *Each function in* $\mathscr{P}'_a(\alpha)$ *maps* $|z|<R$ *onto a convex region where* R *is the smallest positive root of the equation*

$$1 - 2(\alpha - a)r^2 - 4a(1-\alpha)r^3 - (1-2\alpha)r^4 = 0.$$

PROOF. From [3, p. 359] we have that the image of $|z|<r$ under $F(z)$ is convex if $\text{Re}\{zF''(z)/F'(z)+1\}>0$ for $|z|<r$. Now $F(z)\in\mathscr{P}'_a(\alpha)$ implies that $F'(z)\in\mathscr{P}_{2a}(\alpha)$. Hence from Theorem 2 we obtain by letting $F'(z)= P(z)$

$$\begin{aligned} \text{Re}\{zF''(z)/F'(z)+1\} &= \text{Re}\{zP'(z)/P(z)+1\} \geqq 1 - |zP'(z)/P(z)| \\ &\geqq 1 - \frac{(1-\alpha)|z|}{1-|z|^2}\left\{\frac{2a|z|^2+4|z|+2a}{(1-2\alpha)|z|^2+2a(1-\alpha)|z|+1}\right\} \qquad (7) \\ &= \frac{1-2(\alpha-a)|z|^2-4a(1-\alpha)|z|^3-(1-2\alpha)|z|^4}{(1-|z|^2)((1-2\alpha)|z|^2+2a(1-\alpha)|z|+1)}. \end{aligned}$$

Since the denominator of the RHS of (7) is positive for $\alpha \in [0, 1)$ and $|z|<1$ then the fraction will be positive when the numerator is positive. The proof is completed by letting $r=|z|$.

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