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# ON THE LOCATION OF ZEROS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. The paper considers the location of zeros of the equation $(\alpha(t)x')'+\gamma(t)x=0$, $t\in[t_0, t_1]$. The following theorem is proved. Let $[a, a+T]$, $T=na$ ($n$ a positive integer), be a subset of $[t_0, t_1]$. Denote $\omega=\pi/T$. Let the coefficient functions obey the inequality $\int_a^{a+T}\{\gamma(t)-\omega^2\alpha(t)\sin^2(\omega t)\}\,dt>\omega^2\int_a^{a+T}\{\alpha\cos 2\omega t\}\,dt$. Then every solution of this equation will have a zero on $[a, a+T]$. A more general form of this theorem is also proved.

0. **Summary.** This note provides a corollary to Leighton's variational theorem, providing a sufficient condition for the existence of a zero on an interval of given length for a second-order selfadjoint equation.

1. **The selfadjoint linear differential equation.** We consider the equation

$$L(x) = (\alpha(t)x')' + \gamma(t)x = 0, \tag{1}$$

$t\in[t_0, t_1)$, $\alpha(t)\in C^1[t_0, t_1)$, $\gamma(t)\in C[t_0, t_1)$ $('\equiv d/dt)$, where the possibility $t_1=+\infty$ is not excluded.

We wish to find an answer to the following problem. Does every (classical) solution of (1) vanish on every interval of length $T$ $(T<(t_1-t_0))$? This question is not answered completely in this paper, but a sufficient condition is given for the existence of zeros on every subinterval of $[t_0, t_1)$ of length $T$. We shall denote by $\omega$ the number: $\omega=\pi/T$.

THEOREM 1. *Let $[a, a+T]$ be a closed subinterval of $[t_0, t_1)$, where $a=nT$, $n$ an integer. Let the coefficient functions $\alpha(t)$, $\gamma(t)$ obey the inequality*

$$\int_a^{a+T}[\gamma(t) - \omega^2\alpha(t)]\sin^2\omega t\,dt - \omega^2\int_a^{a+T}\alpha(t)\cos(2\omega t)\,dt > 0. \tag{2}$$

*Then every solution of* (1) *will vanish on the interval* $[a, a+T]$.

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PROOF. The inequality (2) states that

$$\int_a^{a+T} [(\gamma(t) - \omega^2\alpha(t))\sin^2 \omega t - \omega^2\alpha(t)\cos 2\omega t]\, dt > 0.$$

Integrating by parts the second term of the integrand, we have

$$\int_a^{a+T} [(\gamma(t) - \omega^2\alpha(t))\sin^2 \omega t + \omega\alpha'(t)\sin(\omega t)\cos(\omega t)]dt > 0. \tag{3}$$

We substitute $u(t)=\sin \omega t$, setting $\omega^2=c/a$, where $c$ and $a$ are any suitable positive constants. Obviously

$$(au')' + cu = 0, \tag{4}$$

and $u(n\pi/\omega)=u((n+1)\pi/\omega)=0$ for any integer $n$. The inequality (3) becomes:

$$\begin{aligned}\int_{n\pi/\omega}^{(n+1)\pi/\omega} &\left\{\left(\gamma(t) - \frac{\alpha(t)c}{a}\right) u^2 + auu'\left(\frac{\alpha(t)}{a}\right)'\right\} dt \\ &= \int_{n\pi/\omega}^{(n+1)\pi/\omega} (u \cdot Lu)\, dt \geqq 0.\end{aligned} \tag{5}$$

(See for example [**2**, Equation 1.16, p. 8] for details of manipulation of equality (5).)

The inequality (5) $\int_{n\pi/\omega}^{(n+1)\pi/\omega} (u \cdot Lu)\, dt \geqq 0$ allows us to apply the classical form of Leighton's variational theorem (see [**1**]), which concludes that every real solution of (1) will vanish on the interval $[a, a+T]$, completing the proof.

COROLLARY 1. *Theorem* 1 *is valid if* (2) *and* (5) *are replaced by*:

$$\gamma(t) - \omega^2\alpha(t) \geqq 0 \tag{2a}$$

*and*

$$\int_a^{a+T} [\alpha'(t)\sin 2\omega t]\, dt > 0 \tag{2b}$$

*on* $[a, a+T]$, *or by the single condition*

$$\int_{n\pi/\omega}^{(n+1)\pi/\omega} (\gamma(t)\sin^2 \omega t)\, dt \geqq \omega^2 \int_{n\pi/\omega}^{(n+1)\pi/\omega} \alpha(t)\cos^2(\omega t)\, dt. \tag{2c}$$

*Note*. (2c) is obtained from (2) after a trigonometric substitution.

COROLLARY 2. *If* $\alpha(t) \in C^1[t_0, \infty)$, $\gamma(t) \in C[t_0, \infty)$ *and condition* (2) *is satisfied* (*or if the equivalent conditions* (2a), (2b) *or the condition* (2c) *is satisfied*), *then solutions of* (1) *are oscillatory*, *and will vanish on any interval*

$[\tilde{t}_1, \tilde{t}_2]$ *of length greater or equal to T, provided* $\tilde{t}_1 \geqq n\pi/\omega \geqq \tilde{t}_0$ *for some integer n.*

EXAMPLE 1. Consider the equation

$$y'' + \left(\nu t^\mu - \frac{1}{2\phi} + K \frac{1+t}{1+\sin t}\right) y = 0, \qquad 0 \leqq t < 3\pi/2,$$

where $\nu > 0$, $\mu \geqq 0$, $\phi > 1$ and $K \geqq 1$. We claim that every solution of this equation will vanish on the interval $[0, \sqrt{2}\pi]$. We choose $\omega = \sqrt{2}/2$, and $T = \pi/\omega = \sqrt{2}\pi$. Using inequality (2c), we compute

$$\int_0^{\sqrt{2}\pi} \left[\left(\nu t^\mu - \frac{1}{2\phi} + K \frac{1+t}{1+\sin t}\right) \sin^2\left(\frac{\sqrt{2}}{2}\right) t\right] dt$$
$$= \sqrt{2} \int_0^{\pi\xi} \left[\nu(\sqrt{2}\xi)^\mu - \frac{1}{2\phi} + K \frac{\sqrt{2}\xi + 1}{\sin(\sqrt{2}\xi) + 1}\right] \sin^2 \xi \, d\xi$$
$$= \sqrt{2} \int_0^{\pi\xi} \left[C_{\nu\mu} - \frac{1}{2\phi} + K \frac{\sqrt{2}\xi + 1}{\sin(\sqrt{2}\xi) + 1}\right] \sin^2 \xi \, d\xi,$$

where $C_{\nu\mu} > 0$, while $(\sqrt{2}\xi + 1)/(\sin(\sqrt{2}\xi) + 1) > \frac{1}{2}$ on the interval $0 < \xi < \pi$. It follows that the inequality (2c) is satisfied (since $K \geqq 1$ and $\phi \geqq 1$), proving our claim.

EXAMPLE 2. We claim that any solution of equation $y'' + K(1/K + \sin t)y = 0$, $t \geqq 0$, where $K$ is a real number, vanishes on every interval of length $\pi$ on the ray $[0, \infty)$. To prove this statement choose $\omega = 1$, and check the inequality (2c). We comment that the oscillatory behavior of this equation is well known. (See for example a paper by Elshin [3].) We observe that in the proof of Theorem 1 we have used the assumption that $a$ and $c$ (used in the comparison equation (4)) were constant only to facilitate the derivation of inequality (5). However our arguments may be modified as follows:

Let $t \cdot \omega(t) = \phi(t)$ be any function of the class $C^1[t_0, \infty)$, such that $\phi'(t) > 0$, $\lim_{t\to\infty} \phi(t) = +\infty$.

We represent $(\phi')^2$ in the form $(\phi')^2(t) = c(t)/a(t)$, and repeat the basic arguments of Theorem 1, as outlined in the proof of Theorem 2.

THEOREM 2. *If there exists a function* $\phi(t) \in C^2[t_0, t_1]$ *such that* $\phi'(t) > 0$ *for all* $t \in [t_0, t_1]$, *and* $\sin \phi(\theta_1) = \sin \phi(\theta_2) = 0$ *for some* $\theta_1, \theta_2 \in [t_0, t_1]$, $\theta_2 > \theta_1 \geqq t_0$, *and such that*

$$(6) \qquad \int_{\theta_1}^{\theta_2} \{[\gamma(t) - \alpha(t)(\phi')^2]\sin^2(\phi(t)) + \phi'\alpha' \sin \phi(t) \cos \phi(t)\} \, dt > 0,$$

*then every solution of* (1) *will vanish on* $[\theta_1, \theta_2]$.

PROOF. We choose a function

$$a(t) = K \exp \int^{t} - \left(\frac{\phi''(\xi)}{\phi'(\xi)}\right) d\xi, \tag{7}$$

where $K$ is a positive constant. Clearly $a(t)$ satisfies the differential equation

$$a' + (\phi''/\phi')a = 0, \tag{8}$$

and choose

$$c(t) = a(t)(\phi')^2(t), \qquad t_0 \leqq t \leqq t_1. \tag{9}$$

It is easily checked that $u(t) = \sin(\phi(t))$ obeys the differential equation,

$$(a(t)u')' + c(t)u = 0 \tag{10}$$

and that

$$u(\theta_1) = u(\theta_2) = 0, \tag{10a}$$

while the inequality (6) can be rewritten as:

$$\int_{\theta_1}^{\theta_2} \left[ \left( \gamma(t) - \frac{c(t)}{a(t)} \alpha(t) \right) u^2 + a(t)uu' \left( \frac{\alpha(t)}{a(t)} \right)' \right] dt > 0.$$

Now Leighton's variational theorem can be applied directly, completing the proof. Some obvious corollaries can be obtained by combining this result with the Sturm-Piccone comparison theorem. (See for example **[2]** for an exposition.)

EXAMPLE 3. We shall use Theorem 2 to demonstrate that the solutions of the equation $y'' + t^{-1}y' + t^{2r-1+\varepsilon}y = 0$, $t \in (1, \infty)$, $r > \varepsilon > 0$, will vanish on every interval of the form: $t \in [(n\pi r)^{1/r}, ((n+1)\pi r)^{1/r}]$, $t > 1$. (It is easy to show that the solutions are oscillatory.)

PROOF. We choose $\phi(t) = r^{-1}t^r$. The original equation can be written in the selfadjoint form $(ty')' + t^{2r+\varepsilon}y = 0$, so that $\alpha(t) = t$. $\gamma(t) = t^{2r}$. Choosing $\theta_1 = (n\pi r)^{1/r}$, $\theta_2 = [(n+1)\pi r]^{1/r}$, we compute

$$\begin{aligned}
&\int_{\theta_1}^{\theta_2} \{[t^{2r+\varepsilon} - t(t^{2(r-1)})]\sin^2(r^{-1}t^r) + \tfrac{1}{2}t^{r-1}\sin(2r^{-1}t^r)\}\, dt \\
&\quad = \mu + \tfrac{1}{2}\int_{\theta_1}^{\theta_2} t^{r-1}\sin(2r^{-1}t^r)\, dt \\
&\quad = \mu + \tfrac{1}{2}r^{-1}\int_{n\pi r}^{(n+1/2)\pi r} \sin(2r^{-1}\xi)\, d\xi \\
&\quad = \mu + \tfrac{1}{4}\int_{2\pi n}^{2\pi(n+1/2)} \sin\eta\, d\eta = \mu,
\end{aligned}$$

where $\mu=\int_{\theta_1}^{\theta_2}(t^{2r+\varepsilon}-t^r)\sin^2(r^{-1}t^r)\,dt>0$. Hence every solution of this equation will vanish on every interval of length $T=((n+1)\pi r)^{1/r}-(n\pi r)^{1/r}$ for all $t>1$, which was to be shown.

## 2. The equation

$$(\alpha(t)x')'+\gamma(t)f(x)=0, \qquad t\geqq t_0. \tag{11}$$

A similar (weaker) result can be obtained more easily for equation (11) or its special case

$$(\alpha(t)x')'+\gamma(t)x^K=0, \qquad t\geqq t_0, \tag{12}$$

where $K$ is an odd integer, and $\alpha(t)\neq 0$. (There is no loss of generality in assuming $\alpha(t)>0$.) $\alpha(t)\in C^2[t_0,t_1)$, $\gamma(t)\in C[t_0,t_1)$. Using the result of this author **[4]**, and putting $u(t)=\sin\omega t$, $\alpha=t_1=n\pi/\omega$, $\beta=t_2=(n+1)\pi/\omega$ (using symbolism of **[4]**), we obtain the following:

COROLLARY. *Let $G(\xi)$ be any function such that $G(0)=0$, and $G(\xi)>0$ if $\xi\neq 0$. Denote $dG/d\xi$ by $g(\xi)$. Let $\omega>0$ be a number such that*

$$\int_{t_1=n\pi/\omega}^{t_2=(n+1)\pi/\omega}[\gamma(t)G(\sin\omega t)-\omega^2\alpha(t)\cos^2\omega t]\,dt>0 \tag{13}$$

*for some integer n. Then any solution $x(t)$ of equation* (12) *will have the property that $|\hat{x}(t)|<(m/K)^{1/K-1}$ for some $t\in[t_1,t_2]$, $t_1=n\pi/\omega$, $t_2=(n+1)\pi/\omega$, where $m=\max_{t\in[t_1,t_2]}(g^2(\sin\omega t)/4G(\sin\omega t))$, provided such maximum exists.*

In the more general case of (11), we easily have a similar result. The inequality (13) with $G(\xi)$ having identical properties on $(n\pi/\omega,(n+1)\pi/\omega)$ implies that every solution $x(t)$ of (11) will have the property that $f'(x(t))<m$ $(f'(x)=df(x)/dx)$, on some subinterval of $(t_1=n\pi/\omega,\ t_2=(n+1)\pi/\omega)$, where as before $m=\max_{t\in[t_1,t_2]}[g^2(\sin\omega t)/4G(\sin\omega t)]$, provided $m$ exists.

EXAMPLE 4. Consider the equation

$$(x^2y')'+(x^2+K(\sin x)/x)y^5=0, \qquad x>\pi,\quad K\leqq 1,$$

which is equivalent to the Emden-Fowler equation perturbed by the $(K(\sin x)/x)y^5$ term.

We claim that all solutions will attain values smaller in absolute value than .7 on every interval of length equal to $\pi^2$. Choosing $G(\xi)=\xi^2$ $(m\equiv 1)$, $\omega=1/\pi$, we compute according to formula (13)

$$\int_{n\pi/\omega}^{(n+1)\pi/\omega} t^2\frac{\sin^2(\omega t)}{4}-\omega^2\left(t^2+K\frac{\sin t}{t}\right)\cos^2(\omega t)\,dt$$
$$=\int_{n\pi^2}^{(n+1)\pi^2}\tau^2\left(\frac{\pi^2}{4}\sin^2\tau-\cos^2\tau\right)-\frac{K}{\pi^2}\left(\frac{\sin(\pi\tau)}{\pi\tau}\right)d\tau.$$

A rough numerical computation shows that for $n \geqq 1$ $(\tau \geqq \pi^2)$, $K \leqq 1$, this integral is positive. Hence $df(y)/dy = 5y^4$ will attain values smaller than $m \equiv 1$, or $|y(x)| < \sqrt[4]{(1/5)} < .7$ on some subinterval of $[n\pi^2, (n+1)\pi^2]$, as required.

Clearly this estimate is valid for the Emden-Fowler equation $y'' + (2/x)y' + y^5 = 0$, $x > \pi$. We remark that a more detailed numerical computation would result in an improved estimate.

## References


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