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NONFACTORIZATION IN SUBSETS OF THE MEASURE ALGEBRA

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ABSTRACT. In this note we unify and simplify some recent results showing the impossibility of factoring in certain convolution subalgebras of the group algebra of a nondiscrete LCAG. A new result is a direct proof of nonfactorization of the classical Hardy spaces, regarded as convolution algebras, on the circle. By considering the ideal of Hilbert-Schmidt operators in the algebra of compact operators on a Hilbert space we illustrate that nonfactorization is not peculiar to convolution.

It is known that the group algebra of a LCAG has the factorization property. The general case is due to Cohen **[5]**, while Rudin **[13]**, **[14]**, solved the problem for locally Euclidean G. In 1939, Salem **[15]** obtained the same result for the circle group.

Recently, **[10]**, **[18]**, **[17]**,[1] attention has been given to the impossibility of factoring by convolution in subalgebras of group algebras. Most of these nonfactorization results are for explicit choices of Reiter's Segal algebras **[12]**. For further details on history and related results see **[8]**, **[9]**.

It may be of interest to keep in mind recent generalizations of Segal algebras due to Burnham **[2]**, **[3]** and Cigler **[4]**.

Our objective is to present simple proofs of some of the known nonfactorization results by exploiting a single idea (Lemma A). We implicitly use the fact that if G is a nondiscrete LCAG [fixed for the remainder of this note], then as a consequence of Theorem 1 in **[6]**, $\exists f \in C_c(G)$ with $\hat{f} \notin L^1(\hat{G})$.

Several of the known nonfactorization results are a simple consequence of

LEMMA A. *Let $(X, \mathscr{F}, \mu)$ be a positive measure space. Let f be a bounded complex-valued function on X with $f \in L^p(\mu)$ for some $p \in (1, \infty)$. If for each*

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[1] This was brought to our attention after an earlier draft of this paper had been accepted for publication. There is some overlap in material and the results in **[17]** are more comprehensive but the techniques of the present note are sufficiently different from those in **[17]** to warrant this note. In particular our result concerning the Hardy spaces is not included in **[17]**.

positive integer n, we can write $f=f_1 \cdots f_n$ (pointwise multiplication) with f_i bounded and $f_i \in L^p(\mu)$, then $f \in L^1(\mu)$.

PROOF. Since f is bounded, $f \in L^t$ for $t \geqq p$. Choose $t=n\geqq p$ and $a_i=1/n$. Since $|f_i|^t \in L^1$ we have by the generalized Hölder inequality,

$$\int_X (|f_1|^t)^{a_1} \cdots (|f_n|^t)^{a_n}\, d\mu \leqq \| \, |f_1|^t \|_1^{a_1} \cdots \| \, |f_n|^t \|_1^{a_n} < \infty.$$

But the left side of the inequality is precisely $\|f\|_1$ so the proof is complete. Here are our nonfactorization results (we write A^2 for $A * A$).

THEOREM B. *If $\hat{A} \not\subset L^1(\hat{G})$ but $\hat{A} \subset L^p(\hat{G})$ for some $p \in (1, \infty)$, then $A^2 \neq A$.*

PROOF. Suppose $A^2=A$. Then for each $g \in A$ and positive integer n we can write $g=g_1 * \cdots * g_n$. Taking Fourier-Stieltjes transforms and applying Lemma A we obtain the contradictory fact $\hat{g} \in L^1(\hat{G})$.

COROLLARY C. *If $p \in (1, \infty)$, then $A_p(G)=\{f \in L^1(G) \mid \hat{f} \in L^p(\hat{G})\}$ fails to factor. More generally, if η is a positive measure on $\hat{G}$, then $A_p(G, \eta)$ fails to factor.*

THEOREM D. *Let $\hat{A} \not\subset L^1(\hat{G})$. If $A \subset L^p(G)$ for some $p \in (1, 2]$ or if $A \subset L^1(G) \cap L^p(G)$ for some $p>1$, then $A^2 \neq A$.*

This is also proved by applying Lemma A, using the Hausdorff-Young theorem and Plancherel's theorem.

COROLLARY E. *$L^1(G) \cap L^p(G)$ fails to factor for all $p \neq 1$.*

COROLLARY F. *Identify G with $[0, 2\pi)$. If $p \in (1, \infty)$, then $H^p(G)$ fails to factor.*

PROOF. By a result of Boas [**1**], L^p and H^p are topologically isomorphic (H^p is given the inherited L^p norm). Corollary E applied to the explicit construction in [**1**] gives the result.

EXAMPLE. Let C=ideal of compact operators on some Hilbert space. Let B=ideal of Hilbert-Schmidt operators (in C) with the Hilbert-Schmidt norm. Then [**3**] B is an A-Segal algebra in C. Moreover $B^2=$ trace class $\subsetneq B$. It would be interesting to know if the (proper) A-Segal algebras of [**2**] fail to factor. An affirmative answer would settle, in particular, the problem for all (proper) Segal algebras: a plausible conjecture is that all proper symmetric Segal algebras [**11**, p. 17] fail to factor.

REMARK. In [**7**], [**16**], it is shown that $M_0(G)$ fails to factor by delicate ideal theoretic arguments. It would be of interest to have a direct proof of this fact within the context of the techniques of this note.

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