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EXISTENCE OF A SEMINORMAL BASIS IN $C[0,1]$

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ABSTRACT. The existence of a basis in $C[0, 1]$ consisting of pairwise orthogonal elements is established.

1. **Introduction.** Let B be a real Banach space. $x\in B$ is orthogonal to $y\in B$, written $x\perp y$, if $\|x+by\|\geqq\|x\|$ for all real numbers b. $x\in B$ is orthogonal to a subset S of B, written $x\perp S$, if $x\perp y$ for all $y\in S$. A basis $\{x_i\}$ of B is normal if, for all $i=1,2,\cdots$, $\|x_i\|=1$ and $x_i\perp[x_1,x_2,\cdots,x_{i-1},x_{i+1},\cdots]$, the closed subspace of B spanned by $x_1,x_2,\cdots$. A basis of B is seminormal if $\|x_i\|=1$ and $x_i\perp x_j$ for all $i\neq j$. Clearly every normal basis is a seminormal basis. The converse of this is not true in general. The problem of existence of a normal basis in $C[0, 1]$ would be solved in the negative if the nonexistence of a seminormal basis in $C[0, 1]$ is proved. However, we will show in this note that $C[0, 1]$ does possess a seminormal basis.

2. **Main result.** Let $a_0=0$, $a_1=1$, and if $j=2^n$, $a_{j+k}=(2k-1)/(2j)$ $(n=0,1,2,\cdots;k=1,2,\cdots,2^n)$. We define a sequence $\{x_i\}$ in $C[0,1]$ as follows:

$$\begin{aligned} x_0(t) &= 1, && t=a_0,\\ &= 0, && t\in\{a_1,a_2\},\\ &= \text{linear}, && \text{for other } t,\\ x_1(t) &= 1, && t=a_1,\\ &= 0, && t\in\{a_0,a_2\},\\ &= \text{linear}, && \text{for other } t,\end{aligned}$$

and, for $j=2^n$ $(n=0,1,\cdots;k=1,\cdots,2^n)$,

$$\begin{aligned} x_{j+k}(t) &= 0, && t\in\{a_0,a_1,\cdots,a_j,b_j\},\\ &= 1, && t\in\{a_{j+1},\cdots,a_{j+k}\},\\ &= -1, && t\in\{a_{j+k+1},\cdots,a_{2j}\}\ (=\varnothing,\ \text{if } k=j),\\ &= \text{linear}, && \text{for the other } t,\end{aligned}$$

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where

$$b_j = a_{2j+1}, \quad \text{if } n \text{ is even,}$$
$$= a_{4j}, \quad \text{if } n \text{ is odd.}$$

THEOREM. *$\{x_i\}$ is a seminormal basis of $C[0, 1]$.*

Obviously, $\|x_i\|=1$. We need only show $x_i \perp x_j$ $(i\neq j)$ and $\{x_i\}$ is a basis of $C[0, 1]$.

3. **Orthogonality.** The fact that $x_i \perp x_j$ $(i\neq j)$ follows from the following:

PROPOSITION. *Let $x, y \in C[0, 1]$. Suppose $\{t \in [0, 1]: |x(t)|=\|x\|\}=\{t_1, \cdots, t_n\}$. Then $x \perp y$ if and only if $y(t_i)x(t_i)y(t_j)x(t_j)\leqq 0$ for some $1\leqq i, j\leqq n$.*

PROOF. The condition is clearly sufficient. On the other hand, suppose $y(t_i)x(t_i)y(t_j)x(t_j)>0$ for all $1\leqq i, j\leqq n$. Then $y(t_i)x(t_i)$ have the same sign, say positive, for all $i=1, 2, \cdots, n$. By the continuity of x and y, there exist positive numbers p and q, and a closed subset F of $[0, 1]$ such that $\{t_1, \cdots, t_n\}\subset F$, $|x(t)|\leqq\|x\|-p$ for $t \in [0, 1]\backslash F$, $|x(t)|\geqq\frac{1}{2}$ for $t \in F$, and $|y(t)|\geqq q>0$ for t in F. Hence $\|x+by\|<\|x\|$ if $b=-(p/(2\|y\|))$.

COROLLARY. *If $x \in C[0, 1]$ is such that $n=1$ in the Proposition, then $x \perp y$ if and only if $y(t_1)=0$.*

4. **$\{x_i\}$ is a basis.** We will show this in two steps:

A. We construct a basis $\{y_i\}$ of $C[0, 1]$ from the Schauder basis $\{z_i\}$ of $C[0, 1]$.

B. We construct $\{x_i\}$ from $\{y_i\}$.

The following method of piecewise construction of new basis in Banach space is important for our purpose here. The proof can be found in **[1**, p. 64].

LEMMA. *Let $\{z_n\}$ be a basis of a Banach space E, $\{m(n)\}$ an increasing sequence of positive integers, $m(0)=0$, and $\{y_n\}$ a sequence in E such that*

(a) $[y_{m(n-1)+1}, \cdots, y_{m(n)}]=[z_{m(n-1)+1}, \cdots, z_{m(n)}]$ $(n=1, 2, \cdots)$, *and*

(b) *there exists a positive constant C such that*

$$\left\| \sum_{i=m(n-1)+1}^{j} c_i y_i \right\| \leqq C \left\| \sum_{i=m(n-1)+1}^{m(n)} c_i y_i \right\|$$

for any sequence $\{c_i\}$ of real numbers, and $m(n-1)+1\leqq j\leqq m(n)$ $(n=1, 2, \cdots)$.

Then $\{y_i\}$ is a basis of E.

4A. Let $\{z_i\}$ be the Schauder basis of $C[0, 1]$, i.e., $z_0(t)=1-t$, $z_1(t)=t$, and, for $j=2^n$ $(n=0, 1, \cdots; k=1, 2, \cdots, 2^n)$,

$$\begin{aligned} z_{j+k}(t) &= 0, & t &\in \{a_0, a_1, \cdots, a_j\}, \\ &= 1, & t &= a_{j+k}, \\ &= \text{linear}, & &\text{for the other } t. \end{aligned}$$

Now define a sequence $\{y_i\}$ as follows: $y_0(t)=x_0(t)$, $y_1(t)=x_1(t)$, and, for $j=2^n$ $(n=0, 1, \cdots; k=2, 3, \cdots, 2^n)$,

$$\begin{aligned} y_{j+1}(t) &= \tfrac{1}{2}(x_{j+1}(t) + x_{2j}(t)), \\ y_{j+k}(t) &= \tfrac{1}{2}(x_{j+k}(t) - x_{j+k-1}(t)). \end{aligned}$$

It is not hard to see the following is true: $[y_0, \cdots, y_3]=[z_0, \cdots, z_3]$, and, for $j=2^{2n}$ $(n=1, 2, \cdots)$,

$$[y_j, \cdots, y_{2j+1}] = [z_j, \cdots, z_{2j+1}], \quad y_i = z_i \ (i = 2(j+1), \cdots, 4j-1).$$

Now let $\{c_i\}$ be any sequence of real numbers. Set $b_i=|c_i|$ $(i=0, 1, 2)$ and $b_3=|c_3+\frac{1}{2}c_0|$, then

$$\begin{aligned} \left\| \sum_{i=0}^{3} c_i y_i \right\| &= \max\{b_k : 0 \leqq k \leqq 3\} \geqq \max\{b_k : 0 \leqq k \leqq j \leqq 3\} \\ &= \left\| \sum_{i=0}^{j} c_i y_i \right\| \qquad (0 \leqq j \leqq 3). \end{aligned}$$

And for $j=2^{2n}$ $(n=1, 2, \cdots)$, set $b_i=|c_i|$ $(i=j, j+1, j+2, \cdots, 2j-2, 2j, 2j+1)$ and $b_{2j-1}=|c_{2j-1}+\frac{1}{2}c_j|$. Then

$$\begin{aligned} \left\| \sum_{k=0}^{j+1} c_{j+k} y_{j+k} \right\| &= \max\{b_{k+j} : 0 \leqq k \leqq j+1\} \\ &\geqq \max\{b_{k+j} : 0 \leqq k \leqq m \leqq j+1\} \\ &= \left\| \sum_{k=0}^{m} c_{j+k} y_{j+k} \right\| \qquad (0 \leqq m \leqq j+1). \end{aligned}$$

Hence by the Lemma cited above, $\{y_i\}$ is a basis of $C[0, 1]$.

4B. From the definition of x_i and y_i, it follows that $x_i=y_i$ $(i=0, 1, 2)$ and $[x_{j+1}, \cdots, x_{2j}]=[y_{j+1}, \cdots, y_{2j}]$ $(j=2^n, n=1, 2, \cdots)$. Again let $\{c_i\}$ be any sequence of real numbers. For $j=2^{2n}$ $(n=1, 2, \cdots)$ and m a positive integer $\leqq j$, let

$$b_1 = \sum_{i=1}^{m} c_{j+i}, \qquad d_1 = \sum_{i=m+1}^{j} c_{j+i},$$

$$b_k = -\sum_{i=1}^{k-1} c_{j+i} + \sum_{i=k}^{m} c_{j+i} \qquad (k = 2, \cdots, m),$$

$$d_k = -\sum_{i=m+1}^{m+k-1} c_{j+i} + \sum_{i=m+k}^{j} c_{j+i} \qquad (k = 2, \cdots, j-m),$$

then

$$\begin{aligned}\left\|\sum_{i=1}^{j} c_{j+i}x_{j+i}\right\| &= \max\{|b_1 + d_1|, |b_2 + d_1|, \cdots, |b_m + d_1|,\\ &\qquad |-b_1 + d_1|, |-b_1 + d_2|, \cdots, |-b_1 + d_{j-m}|\}\\ &\geqq \max\{|b_1 + d_1|, |b_2 + d_1|, \cdots, |b_m + d_1|, |-b_1 + d_1|\}\\ &\geqq \inf_r\{\max\{|b_1 + r|, |b_2 + r|, \cdots, |b_m + r|, |-b_1 + r|\}\}\\ &= \tfrac{1}{2}(\max\{|b_1|, \cdots, |b_m|\} + |B|) \geqq \tfrac{1}{2}\max\{|b_1|, \cdots, |b_m|\}\\ &= \tfrac{1}{2}\left\|\sum_{i=1}^{m} c_{j+i}x_{j+i}\right\|,\end{aligned}$$

where $B = \max\{b_1, \cdots, b_m\}$ if $\max\{|b_i| : 1 \leqq i \leqq m\} = -b_k$ for some k, $1 \leqq k \leqq m$; or $B = \min\{b_1, \cdots, b_m\}$ if $\max\{|b_i| : 1 \leqq i \leqq m\} = b_k$ for some k, $1 \leqq k \leqq m$. This completes the proof.

Reference

1. I. Singer, *Bases in Banach spaces*. I, Springer, Berlin and New York, 1970.

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