PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, No. 1, September 1972

SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

ON A THEOREM OF RUDIN

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ABSTRACT. We give short proofs of a theorem of Rudin about polynomial approximation in R^{2+n} and a corollary of this theorem which says that any function algebra on [0, 1] generated by one complex-valued function and n real functions is all continuous functions. At the same time our proof shows that both results hold with n replaced by an arbitrary index set Λ.

Denote the points of $R^2 \times R^\Lambda$ by (z, t). $C(K)$ denotes all continuous functions on K.

THEOREM 1. *Let K be a compact subset of $R^2 \times R^\Lambda$ such that $K_t = \{z \mid (z, t) \in K\}$ does not separate the plane for any t. If $f \in C(K)$ and $f_t(z) = f(z, t)$ is analytic at every interior point of K_t then f can be approximated uniformly on K by polynomials in z and t_a $(a \in \Lambda)$.*

PROOF. Let A be the function algebra on K generated by z and the t_a. Let μ be an extreme point of ball$(A^\perp)$. Since the closed support of μ is a set of antisymmetry, μ is concentrated on some $K_t \times \{t\}$. But by Mergelyan's theorem f can be approximated uniformly there by polynomials in z, and so is annihilated of μ. Thus $f \in A$.

The following corollary is immediate from the above but the direct proof is very short.

THEOREM 2. *Let K be a compact subset of the line. If $f \in C(K)$ and u_a $(a \in \Lambda)$ are real-valued functions in $C(K)$ such that f and the u_a separate the points of K then the function algebra A on K generated by f and the u_a is $C(K)$.*

PROOF. Again let μ be an extreme point of ball$(A^\perp)$. Each u_a must be constant on S, the closed support of μ. Thus $f|S$ is a homeomorphism of a compact subset of the line into the plane. It is well known that $f(S)$

Received by the editors October 11, 1971.
AMS 1970 *subject classifications*. Primary 46J10; Secondary 41A10.

cannot separate the plane so Mergelyan's theorem shows $\mu=0$ and thus $A=C(K)$.

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