INTEGRAL AS A CERTAIN TYPE OF A POSITIVE DEFINITE FUNCTION

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Abstract. The integral with respect to an $H^*$-algebra valued measure on a compact Hausdorff space $S$ is characterized as a certain type of positive definite function on the space $C(S)$ of continuous functions on $S$.

1. Let $A$ be a proper $H^*$-algebra, let $\tau(A)$ be its trace-class \cite{5} and let $\mu$ be a $\tau(A)$-valued measure defined on the Borel sets of a compact Hausdorff space $S$. One can define the integral $I(f) = \int f \, d\mu$ on the algebra $B = C(S)$ of all continuous complex-valued functions $f$ on $S$ the way it was done in \cite[p. 120]{8}. Also one can show that the integral $I$ behaves as a positive definite function on $B$. In fact $I$ is a positive $A$-functional on $B$ in the sense of \cite{9}, i.e. $\sum_{i,j} a_i^{*} I(f_i f_j) a_j \geq 0$ for all $\{a_1, \cdots, a_n\} \subseteq A$ and $\{f_1, \cdots, f_n\} \subseteq B$. In this note we shall show that this property can be used to characterize the integral $I$.

This paper can be considered as a continuation of \cite{9}. We shall use the same terminology.

2. Theorem 1 below constitutes the main result of the paper. It can be considered as a generalization of Bochner theorem \cite{8}. Note that unlike Theorem VI.7.3 of \cite{2} we do not make any assumptions about continuity of the mapping $I$.

Theorem 1. Let $A$, $S$ and $B = C(S)$ be as above. If $\mu$ is a positive Borel $\tau(A)$-valued measure on $S$ then $I(f) = \int f \, d\mu$ is a positive $A$-functional on $B$ (see the definition before Lemma 1 in \cite{9}). Conversely each positive $A$-functional $I$ on $B$ is of the form $I(f) = \int f \, d\mu$ for some positive $\tau(A)$-valued measure $\mu$ on $S$.

Proof. The first part of Theorem 1 is established in the way the first part of the theorem on p. 120 of \cite{8} was proven (here again one can show...
that a linear mapping \( I: B \to \tau(A) \) is a positive \( A \)-functional if

\[
\text{tr}
\left(
\sum_{i,j} a_i^* I(f_i,f_j) a_j
\right)
\geq 0
\quad \text{for all } \{a_i\} \subset A, \{f_i\} \subset B.
\]

Now let \( I \) be a positive \( A \)-functional on \( B \). Then [9, Theorem 1] there exists a Hilbert module \( H \), a \( * \)-representation \( f \to T_f \) of \( B \) by continuous \( A \)-linear operators on \( H \) and \( x \in H \) such that \( I(f) = (x, T_f x) \) for all \( f \in B \).

Now note that \( H \) is a Hilbert space under the scalar product \( [x, y] = \text{tr}(y, x) \) and each \( T_f \) is a bounded linear operator on \( H \) (see [6]). Thus one can apply the Proposition II of subsection 4 of §17 of [4]: there exists a spectral measure \( P: \Delta \to P_\Delta \) on \( \mathcal{S} \) such that \( T_f = \int_{\mathcal{S}} f(s) \, dP_s \) for all \( f \in B \) (note that the space \( \mathfrak{M} \) of maximal ideals of \( B \) is homeomorphic to \( \mathcal{S} \) [3, 19C–D]; see also the example after Theorem 3 in §11 of [4]).

From the fact that each \( P_\Delta \) commutes with each operator \( Q \) such that \( QT_f = T_f Q \), \( f \in B \), we conclude that the operators \( P_\Delta \) are \( A \)-linear [6, Definition 4]. Thus the mapping \( \Delta \to P_\Delta \) is a generalized spectral measure [8, p. 118] (also [7, p. 149]) defined on Borel sets of \( \mathcal{S} \). If we define the measure \( \mu \) by \( \mu_\Delta = (x, P_\Delta x) \) then one can readily verify that \( I(f) = (x, T_f x) = (x, \int x(s) \, dP_s x) = \int f(s) \, d(x, P_s x) = \int f(s) \, d\mu(s) \).

3. In a similar manner one can prove also the following two theorems (here also we use Theorem 1 of [9] and the Proposition II in subsection 4 of §17 of [4]).

**Theorem 2.** For each \( * \)-representation \( x \to T_x \), by \( A \)-linear operators, of a commutative \( B^* \)-algebra \( B \), having an identity, there exists a generalized spectral measure \( \Delta \to P_\Delta \) on the space \( \mathfrak{M} \) of maximal ideals of \( B \) such that \( T_x = \int_{\mathfrak{M}} x(M) \, dP_M \) for each \( x \in B \) (we use the terminology of [4] here).

**Theorem 3.** Each positive \( A \)-functional \( \rho \) on the algebra \( B \) is of the form \( \rho(x) = \int_{\mathfrak{M}} x(M) \, d\mu(M) \) for some positive \( \tau(A) \)-valued Borel measure \( \mu \) on \( \mathfrak{M} \).

**References**


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