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# INTEGRAL AS A CERTAIN TYPE OF A POSITIVE DEFINITE FUNCTION

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ABSTRACT. The integral with respect to an $H^*$-algebra valued measure on a compact Hausdorff space $S$ is characterized as a certain type of positive definite function on the space $C(S)$ of continuous functions on $S$.

1. Let $A$ be a proper $H^*$-algebra, let $\tau(A)$ be its trace-class [**5**] and let $\mu$ be a $\tau(A)$-valued measure defined on the Borel sets of a compact Hausdorff space $S$. One can define the integral $I(f)=\int f\,d\mu$ on the algebra $B=C(S)$ of all continuous complex-valued functions $f$ on $S$ the way it was done in [**8**, p. 120]. Also one can show that the integral $I$ behaves as a positive definite function on $B$. In fact $I$ is a positive $A$-functional on $B$ in the sense of [**9**], i.e. $\sum_{i,j} a_i^* I(\bar{f}_i f_j) a_j \geqq 0$ for all $\{a_1, \cdots, a_n\} \subset A$ and $\{f_1, \cdots, f_n\} \subset B$. In this note we shall show that this property can be used to characterize the integral $I$.

This paper can be considered as a continuation of [**9**]. We shall use the same terminology.

2. Theorem 1 below constitutes the main result of the paper. It can be considered as a generalization of Bochner theorem [**8**]. Note that unlike Theorem VI.7.3 of [**2**] we do not make any assumptions about continuity of the mapping $I$.

THEOREM 1. *Let $A$, $S$ and $B=C(S)$ be as above. If $\mu$ is a positive Borel $\tau(A)$-valued measure on $S$ then $I(f)=\int f\,d\mu$ is a positive $A$-functional on $B$ (see the definition before Lemma 1 in* [**9**]). *Conversely each positive $A$-functional $I$ on $B$ is of the form $I(f)=\int f\,d\mu$ for some positive $\tau(A)$-valued measure $\mu$ on $S$.*

PROOF. The first part of Theorem 1 is established in the way the first part of the theorem on p. 120 of [**8**] was proven (here again one can show

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that a linear mapping $I: B \to \tau(A)$ is a positive $A$-functional if

$$\operatorname{tr}\left(\sum_{i,j} a_i^* I(\bar{f}_i f_j) a_j\right) \geqq 0 \quad \text{for all } \{a_i\} \subset A,\ \{f_i\} \subset B).$$

Now let $I$ be a positive $A$-functional on $B$. Then [**9**, Theorem 1] there exists a Hilbert module $H$, *a* *-representation $f \to T_f$ of $B$ by continuous $A$-linear operators on $H$ and $x \in H$ such that $I(f) = (x, T_f x)$ for all $f \in B$.

Now note that $H$ is a Hilbert space under the scalar product $[x, y] = \operatorname{tr}(y, x)$ and each $T_f$ is a bounded linear operator on $H$ (see [**6**]). Thus one can apply the Proposition II of subsection 4 of §17 of [**4**]: there exists a spectral measure $P: \Delta \to P_\Delta$ on $S$ such that $T_f = \int_S f(s)\, dP_s$ for all $f \in B$ (note that the space $\mathfrak{M}$ of maximal ideals of $B$ is homeomorphic to $S$ [**3**, 19C–D]; see also the example after Theorem 3 in §11 of [**4**]).

From the fact that each $P_\Delta$ commutes with each operator $Q$ such that $QT_f = T_f Q$, $f \in B$, we conclude that the operators $P_\Delta$ are $A$-linear [**6**, Definition 4]. Thus the mapping $\Delta \to P_\Delta$ is a generalized spectral measure [**8**, p. 118] (also [**7**, p. 149]) defined on Borel sets of $S$. If we define the measure $\mu$ by $\mu\Delta = (x, P_\Delta x)$ then one can readily verify that $I(f) = (x, T_f x) = (x, (\int f(s)\, dP_s)x) = \int f(s)\, d(x, P_s x) = \int f(s)\, d\mu(s)$.

3. In a similar manner one can prove also the following two theorems (here also we use Theorem 1 of [**9**] and the Proposition II in subsection 4 of §17 of [**4**]).

THEOREM 2. *For each *-representation $x \to T_x$, by A-linear operators, of a commutative $B^*$-algebra B, having an identity, there exists a generalized spectral measure $\Delta \to P_\Delta$ on the space $\mathfrak{M}$ of maximal ideals of B such that $T_x = \int_{\mathfrak{M}} x(M)\, dP_M$ for each $x \in B$ (we use the terminology of* [**4**] *here).*

THEOREM 3. *Each positive A-functional p on the algebra B is of the form $p(x) = \int_{\mathfrak{M}} x(M)\, d\mu(M)$ for some positive $\tau(A)$-valued Borel measure $\mu$ on $\mathfrak{M}$.*

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