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A FIXED POINT THEOREM FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. Let K be a subset of a Banach space X. A mapping $F:K\to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_i\}$ of real numbers with $k_i\to 1$ as $i\to\infty$ such that $\|F^ix-F^iy\|\leqq k_i\|x-y\|$, $x, y\in K$. It is proved that if K is a nonempty, closed, convex, and bounded subset of a uniformly convex Banach space, and if $F:K\to K$ is asymptotically nonexpansive, then F has a fixed point. This result generalizes a fixed point theorem for nonexpansive mappings proved independently by F. E. Browder, D. Göhde, and W. A. Kirk.

In 1965, F. E. Browder [**1**] and D. Göhde [**4**] independently proved that every nonexpansive self-mapping of a closed convex and bounded subset of a uniformly convex Banach space has a fixed point. This result was also obtained by W. A. Kirk [**5**], under assumptions slightly weaker in a technical sense, and another proof, more geometric and elementary in nature, has recently been given by K. Goebel [3]. Our purpose here is to extend Browder's result to a more general class of transformations which we shall call "asymptotically nonexpansive" mappings.

A Banach space X is called *uniformly convex* (Clarkson [**2**]) if for each $\varepsilon>0$ there is a $\delta(\varepsilon)>0$ such that if $\|x\|=\|y\|=1$ then $\|(x+y)/2\|\leqq 1-\delta(\varepsilon)$. In such a space, it is easily seen that the inequalities $\|x\|\leqq d$, $\|y\|\leqq d$, $\|x-y\|\geqq\varepsilon$ imply $\|(x+y)/2\|\leqq(1-\delta(\varepsilon/d))d$. Furthermore, the function $\delta:(0, 2]\to(0, 1]$ may be assumed to be increasing.

DEFINITION. Let K be a subset of a Banach space X. A transformation $F:K\to K$ is said to be *nonexpansive* if for arbitrary $x, y\in K$,

$$\|Fx - Fy\| \leqq \|x - y\|.$$

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More generally, F is said to be *asymptotically nonexpansive* if for each $x, y \in K$,

$$\|F^i x - F^i y\| \leqq k_i \|x - y\|$$

where $\{k_i\}$ is a sequence of real numbers such that $\lim_{i\to\infty} k_i = 1$.

It is obvious that for asymptotically nonexpansive mappings it may be assumed that $k_i \geqq 1$ and that $k_{i+1} \leqq k_i$ for $i = 1, 2, \cdots$, so throughout the paper we shall always assume this to be the case.

Our principal result is the following generalization of Browder's theorem of **[1]**.

THEOREM 1. *Let K be a nonempty, closed, convex and bounded subset of a uniformly convex Banach space X, and let $F: K \to K$ be asymptotically nonexpansive. Then F has a fixed point.*

PROOF. For each $x \in K$ and $r > 0$ let $S(x, r)$ denote the spherical ball centered at x with radius r. Let $y \in K$ be fixed, and let the set R_y consist of those numbers ρ for which there exists an integer k such that

$$K \cap \left(\bigcap_{i=k}^{\infty} S(F^i y, \rho) \right) \neq \varnothing .$$

If d is the diameter of K then $d \in R_y$, so $R_y \neq \varnothing$. Let $\rho_0 = \text{g.l.b. } R_y$, and for each $\varepsilon > 0$ define (cf. [**6**, p. 411]) $C_\varepsilon = \bigcup_{k=1}^{\infty} (\bigcap_{i=k}^{\infty} S(F^i y, \rho_0 + \varepsilon))$. Thus for each $\varepsilon > 0$ the sets $C_\varepsilon \cap K$ are nonempty and convex, so reflexivity of X implies that

$$C = \bigcap_{\varepsilon > 0} (\bar{C}_\varepsilon \cap K) \neq \varnothing .$$

Note that for $x \in C$ and $\eta > 0$ there exists an integer N such that if $i \geqq N$, $\|x - F^i y\| \leqq \rho_0 + \eta$.

Now let $x \in C$ and suppose the sequence $\{F^n x\}$ does not converge to x (i.e., suppose $Fx \neq x$). Then there exists $\varepsilon > 0$ and a subsequence $\{F^{n_i} x\}$ of $\{F^n x\}$ such that $\|F^{n_i} x - x\| \geqq \varepsilon$, $i = 1, 2, \cdots$. For $m > n$,

$$\|F^n x - F^m x\| \leqq k_n \|x - F^{m-n} x\|,$$

where k_n is the Lipschitz constant for F^n obtained from the definition of asymptotic nonexpansiveness. Assume $\rho_0 > 0$ and choose $\alpha > 0$ so that $(1 - \delta(\varepsilon/(\rho_0 + \alpha)))(\rho_0 + \alpha) < \rho_0$. Select n so that $\|x - F^n x\| \geqq \varepsilon$ and also so that $k_n(\rho_0 + \alpha/2) \leqq \rho_0 + \alpha$. If $N \geqq n$ is sufficiently large, then $m > N$ implies

$$\|x - F^{m-n} y\| \leqq \rho_0 + \alpha/2,$$

and we have

$$\|F^n x - F^m y\| \leqq k_n \|x - F^{m-n} y\| \leqq \rho_0 + \alpha,$$
$$\|x - F^m y\| \leqq \rho_0 + \alpha.$$

Thus by uniform convexity of X, if $m>N$,

$$\|(x + F^n x)/2 - F^m y\| \leqq (1 - \delta(\varepsilon/(\rho_0 + \alpha)))(\rho_0 + \alpha) < \rho_0,$$

and this contradicts the definition of ρ_0. Hence we conclude $\rho_0=0$ or $Fx=x$. But $\rho_0=0$ implies $\{F^n y\}$ is a Cauchy sequence yielding $F^n y \to x = Fx$ as $n\to\infty$. Therefore the set C consists of a single point which is fixed under F.

THEOREM 2. *Under the same assumptions as in Theorem* 1, *the set Y of fixed points of F is closed and convex.*

PROOF. Closedness of Y is obvious. To show convexity it is sufficient to prove that $z=(x+y)/2 \in Y$ for all $x, y \in Y$. We have

$$\|F^i z - x\| = \|F^i z - F^i x\| \leqq k_i \|z - x\| = \tfrac{1}{2}k_i \|x - y\|,$$
$$\|F^i z - y\| = \|F^i z - F^i y\| \leqq k_i \|z - y\| = \tfrac{1}{2}k_i \|x - y\|.$$

Thus

$$\|z - F^i z\| \leqq \tfrac{1}{2}(1 - \delta(2/k_i))k_i \|x - y\|$$

and hence

$$z = \lim_{i\to\infty} F^i z = \lim_{i\to\infty} F^{i+1} z = F\left(\lim_{i\to\infty} F^i z\right) = Fz.$$

The following theorem shows that in Theorem 1 it need only be assumed that F is "eventually asymptotically nonexpansive".

THEOREM 3. *Suppose K is a nonempty, closed, bounded and convex subset of a uniformly convex Banach space X and suppose $F:K\to K$ is an arbitrary* (*even noncontinuous*) *transformation such that for some integer n,*

$$\|F^i x - F^i y\| \leqq k_i \|x - y\|, \qquad i \geqq n,$$

where $\lim_{i\to\infty} k_i=1$. *Then F has a fixed point.*

PROOF. The transformation $G=F^n$ is asymptotically nonexpansive so it has a nonempty closed and convex fixed point set Y. If $x \in Y$ then $Fx=FGx=F^{n+1}x=GFx$ and thus $F: Y\to Y$. Moreover, $F=F^{pn+1}$ on Y for $p=1, 2, \cdots$. Hence

$$\|Fx - Fy\| = \|F^{pn+1}x - F^{pn+1}y\| \leqq k_{pn+1} \|x - y\|, \qquad x, y \in Y.$$

This implies that $\|Fx-Fy\|\leqq\|x-y\|$, $x, y \in Y$, and according to the fixed point theorem for nonexpansive mappings (Browder [1]), F has a fixed point in Y.

Finally we show that the class of asymptotically nonexpansive mappings is wider than the class of nonexpansive mappings.

EXAMPLE. Let B denote the unit ball in the Hilbert space l^2 and let F be defined as follows:

$$F:(x_1, x_2, x_3, \cdots) \to (0, x_1^2, A_2x_2, A_3x_3, \cdots)$$

where A_i is a sequence of numbers such that $0<A_i<1$ and $\prod_{i=2}^{\infty} A_i=\frac{1}{2}$. Then F is lipschitzian and $\|Fx-Fy\| \leqq 2\ \|x-y\|$, $x, y \in B$; and moreover, $\|F^ix-F^iy\| \leqq 2 \prod_{j=2}^{i} A_j\|x-y\|$ for $i=2, 3, \cdots$. Thus

$$\lim_{i\to\infty} k_i = \lim_{i\to\infty} 2\prod_{j=2}^{i} A_j = 1.$$

Clearly the transformation F is not nonexpansive.

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