A FIXED POINT THEOREM FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

K. GOEBEL\(^1\) AND W. A. KIRK\(^2\)

Abstract. Let \(K\) be a subset of a Banach space \(X\). A mapping \(F:K \rightarrow K\) is said to be asymptotically nonexpansive if there exists a sequence \(\{k_i\}\) of real numbers with \(k_i \to 1\) as \(i \to \infty\) such that
\[
\|F(x) - F(y)\| \leq k_i \|x - y\|, \quad x, y \in K.
\]
It is proved that if \(K\) is a non-empty, closed, convex, and bounded subset of a uniformly convex Banach space, and if \(F:K \rightarrow K\) is asymptotically nonexpansive, then \(F\) has a fixed point. This result generalizes a fixed point theorem for nonexpansive mappings proved independently by F. E. Browder, D. Göhde, and W. A. Kirk.

In 1965, F. E. Browder [1] and D. Göhde [4] independently proved that every nonexpansive self-mapping of a closed convex and bounded subset of a uniformly convex Banach space has a fixed point. This result was also obtained by W. A. Kirk [5], under assumptions slightly weaker in a technical sense, and another proof, more geometric and elementary in nature, has recently been given by K. Goebel [3]. Our purpose here is to extend Browder's result to a more general class of transformations which we shall call "asymptotically nonexpansive" mappings.

A Banach space \(X\) is called uniformly convex (Clarkson [2]) if for each \(\epsilon > 0\) there is a \(\delta(\epsilon) > 0\) such that if \(\|x\| = \|y\| = 1\) then
\[
\|\frac{x+y}{2}\| \leq 1 - \delta(\epsilon).
\]
In such a space, it is easily seen that the inequalities \(\|x\| \leq d, \|y\| \leq d, \|x-y\| \geq \epsilon\) imply \(\|\frac{x+y}{2}\| \leq (1 - \delta(\epsilon/d))d\). Furthermore, the function \(\delta: (0, 2] \to (0, 1]\) may be assumed to be increasing.

Definition. Let \(K\) be a subset of a Banach space \(X\). A transformation \(F:K \rightarrow K\) is said to be nonexpansive if for arbitrary \(x, y \in K\),
\[
\|F(x) - F(y)\| \leq \|x - y\|.
\]

Received by the editors November 15, 1971.

AMS 1969 subject classifications. Primary 4785.

Key words and phrases. Fixed point theorem, nonexpansive mapping, asymptotically nonexpansive mapping, uniformly convex Banach space, lipschitzian mapping.

\(^1\) Research supported by a Kosciuszko Foundation grant while the author was at the University of Iowa.

\(^2\) Research supported by National Science Foundation grant GP-18045.

\(\star\) American Mathematical Society 1972
More generally, $F$ is said to be asymptotically nonexpansive if for each $x, y \in K$,
\[ \| F^i x - F^i y \| \leq k_i \| x - y \| \]
where $\{k_i\}$ is a sequence of real numbers such that $\lim_{i \to \infty} k_i = 1$.

It is obvious that for asymptotically nonexpansive mappings it may be assumed that $k_i \geq 1$ and that $k_{i+1} \leq k_i$ for $i = 1, 2, \ldots$, so throughout the paper we shall always assume this to be the case.

Our principal result is the following generalization of Browder's theorem of [1].

**Theorem 1.** Let $K$ be a nonempty, closed, convex and bounded subset of a uniformly convex Banach space $X$, and let $F : K \to K$ be asymptotically nonexpansive. Then $F$ has a fixed point.

**Proof.** For each $x \in K$ and $r > 0$ let $S(x, r)$ denote the spherical ball centered at $x$ with radius $r$. Let $y \in K$ be fixed, and let the set $R_y$ consist of those numbers $\rho$ for which there exists an integer $k$ such that
\[ K \cap \left( \bigcap_{i=k}^{\infty} S(F^i y, \rho) \right) \neq \emptyset. \]

If $d$ is the diameter of $K$ then $d \in R_y$, so $R_y \neq \emptyset$. Let $\rho_0 = \inf R_y$, and for each $\varepsilon > 0$ define (cf. [6, p. 411])
\[ C_\varepsilon = \bigcup_{k} \bigcap_{\rho=\rho_0+\varepsilon} \left( S(F^i y, \rho_0+\varepsilon) \right). \]
Thus for each $\varepsilon > 0$ the sets $C_\varepsilon \cap K$ are nonempty and convex, so reflexivity of $X$ implies that
\[ C = \bigcap_{\varepsilon > 0} (C_\varepsilon \cap K) \neq \emptyset. \]

Note that for $x \in C$ and $\eta > 0$ there exists an integer $N$ such that if $i \geq N$, $\| x - F^i y \| \leq \rho_0 + \eta$.

Now let $x \in C$ and suppose the sequence $\{F^n x\}$ does not converge to $x$ (i.e., suppose $F x \neq x$). Then there exists $\varepsilon > 0$ and a subsequence $\{F^{n_k} x\}$ of $\{F^n x\}$ such that $\| F^{n_k} x - x \| \geq \varepsilon, i = 1, 2, \ldots$. For $m > n$,
\[ \| F^{n_k} x - F^{m} x \| \leq k_n \| x - F^{m-n} x \|, \]
where $k_n$ is the Lipschitz constant for $F^n$ obtained from the definition of asymptotic nonexpansiveness. Assume $\rho_0 > 0$ and choose $\alpha > 0$ so that $(1 - \delta(\varepsilon/(\rho_0 + \alpha))) (\rho_0 + \alpha) < \rho_0$. Select $n$ so that $\| x - F^n x \| \geq \varepsilon$ and also so that $k_n (\rho_0 + \alpha/2) \leq \rho_0 + \alpha$. If $N \geq n$ is sufficiently large, then $m > N$ implies
\[ \| x - F^{m-n} x \| \leq \rho_0 + \alpha/2, \]
and we have
\[ \| F^n x - F^{m} x \| \leq k_n \| x - F^{m-n} x \| \leq \rho_0 + \alpha, \]
\[ \| x - F^{m} x \| \leq \rho_0 + \alpha. \]
Thus by uniform convexity of $X$, if $m > N$,
\[
\|(x + F^nx)/2 - F^my\| \leq (1 - \delta(\varepsilon/(\rho_0 + \delta)))\varepsilon < \rho_0,
\]
and this contradicts the definition of $\rho_0$. Hence we conclude $\rho_0 = 0$ or $F^nx = x$. But $\rho_0 = 0$ implies $\{F^ny\}$ is a Cauchy sequence yielding $F^ny \to x = Fx$ as $n \to \infty$. Therefore the set $C$ consists of a single point which is fixed under $F$.

Theorem 2. Under the same assumptions as in Theorem 1, the set $Y$ of fixed points of $F$ is closed and convex.

Proof. Closedness of $Y$ is obvious. To show convexity it is sufficient to prove that $z = (x + y)/2 \in Y$ for all $x, y \in Y$. We have
\[
\|Fiz - x\| = \|Fiz - F^ix\| \leq k_i \|z - x\| = \frac{1}{2} k_i \|x - y\|,
\]
\[
\|Fiz - y\| = \|Fiz - F^iy\| \leq k_i \|z - y\| = \frac{1}{2} k_i \|x - y\|.
\]
Thus
\[
\|z - Fiz\| \leq \frac{1}{2} (1 - \delta(2/k_i)) k_i \|x - y\|
\]
and hence
\[
z = \lim_{i \to \infty} Fiz = \lim_{i \to \infty} F^{i+1}z = F \left( \lim_{i \to \infty} Fiz \right) = Fz.
\]

The following theorem shows that in Theorem 1 it need only be assumed that $F$ is "eventually asymptotically nonexpansive".

Theorem 3. Suppose $K$ is a nonempty, closed, bounded and convex subset of a uniformly convex Banach space $X$ and suppose $F: K \to K$ is an arbitrary (even noncontinuous) transformation such that for some integer $n$,
\[
\|F^nx - F^ny\| \leq k_i \|x - y\|, \quad i \geq n,
\]
where $\lim_{n \to \infty} k_i = 1$. Then $F$ has a fixed point.

Proof. The transformation $G = F^o$ is asymptotically nonexpansive so it has a nonempty closed and convex fixed point set $Y$. If $x \in Y$ then $Fx = F^nx = F^{n+1}x = GFx$ and thus $F: Y \to Y$. Moreover, $F = F^{pn+1}$ on $Y$ for $p = 1, 2, \cdots$. Hence
\[
\|Fx - Fy\| = \|F^{pn+1}x - F^{pn+1}y\| \leq k_{pn+1} \|x - y\|, \quad x, y \in Y.
\]
This implies that $\|Fx - Fy\| \leq \|x - y\|$, $x, y \in Y$, and according to the fixed point theorem for nonexpansive mappings (Browder [1]), $F$ has a fixed point in $Y$.

Finally we show that the class of asymptotically nonexpansive mappings is wider than the class of nonexpansive mappings.
Example. Let $B$ denote the unit ball in the Hilbert space $l^2$ and let $F$ be defined as follows:

$$F:(x_1, x_2, x_3, \cdots) \rightarrow (0, x_1^2, A_2 x_2, A_3 x_3, \cdots)$$

where $A_i$ is a sequence of numbers such that $0 < A_i < 1$ and $\prod_{i=2}^\infty A_i = \frac{1}{2}$. Then $F$ is Lipschitzian and $\|Fx - Fy\| \leq 2 \|x - y\|$, $x, y \in B$; and moreover, $\|Fx - F' y\| \leq 2 \prod_{i=2}^\infty A_j \|x - y\|$ for $i = 2, 3, \cdots$. Thus

$$\lim_{i \rightarrow \infty} k_i = \lim_{i \rightarrow \infty} 2 \prod_{j=2}^i A_j = 1.$$ 

Clearly the transformation $F$ is not nonexpansive.

References