PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, No. 1, September 1972

A COUNTEREXAMPLE TO AN ANALOGUE OF ARTIN'S CONJECTURE

P. J. WEINBERGER

ABSTRACT. I construct a counterexample to a conjecture of Larry Goldstein on the density of primes which split completely in none of a set of algebraic number fields. The fields used are all Abelian over the rationals.

1. **Introduction.** Let S be a set of rational primes, and for each $p \in S$ let L_p be a finite dimensional normal extension of the field of rational numbers Q. Let T be the set of those natural numbers divisible only by primes of S, together with one. For each $k \in T$ let L_k be the compositum of those L_p with $p|k$, $p \in S$. Take $L_1 = Q$. Let $n(k)$ be the degree of L_k over Q. Let Δ be the natural density of those rational primes which split completely (into distinct factors) in none of the fields L_p, for all $p \in S$. In ([1], [2]) it is conjectured that if

$$\sum_{k \in T} \frac{\mu^2(k)}{n(k)} < \infty, \tag{1}$$

then

$$\Delta = \sum_{k \in T} \frac{\mu(k)}{n(k)}. \tag{2}$$

This conjecture is known to be true in the cases of finite S, and in the case when $L_p \supset Q(\zeta_{p^2})$ for every prime p, where ζ_j denotes a primitive jth root of one. However, the example constructed below shows that the conjecture is false. This counterexample has S as the set of all primes, and also satisfies

$$\lim_{p \to \infty} \frac{n(p)}{\log(\operatorname{disc}(L_p))} = 0, \tag{3}$$

the condition of the Brauer-Siegel theorem.

2. In this section consider a fixed odd prime p. Let $n > 1$ be an integer and let $m = p^{2^n} - 1$. Then $\deg(Q(\zeta_m)) = \phi(m)$ and by a well-known result on cyclotomic fields, p is unramified in $Q(\zeta_m)$. Since $Q(\zeta_m)$ is Abelian we may unambiguously speak of the decomposition field, k_n, of p in $Q(\zeta_m)$. The

Received by the editors November 23, 1971.
AMS 1970 *subject classifications*. Primary 12A35, 12A75; Secondary 12A65.

above remarks establish the following:

$$\begin{aligned} &k_n \text{ is Abelian, in particular, } k_n \text{ is normal;} \\ (4)\qquad &p \text{ splits completely in } k_n; \\ &\deg(k_n) = \phi(m)/2^n. \end{aligned}$$

Further, we have

$$\deg(k_n) = \phi(m)/2^n \gg \frac{\pi(m)}{2^n} \gg \frac{m}{2^n \log m} \gg \frac{p^{2^n}}{2^{2n}\log p},$$

so

$$\lim_{n\to\infty} \deg(k_n) = \infty. \tag{5}$$

The proof of the following lemma occupies the rest of this section.

LEMMA.

$$\lim_{n\to\infty} \frac{\deg(k_n)}{\log(\operatorname{disc}(k_n))} = 0. \tag{6}$$

PROOF. The proof requires some standard notions from class field theory for absolutely Abelian fields. A concise summary may be found in [3, pp. 4–6].

For any positive integer j let $c(j)$ denote the multiplicative group of reduced residue classes modulo j. Then $Q(\zeta_m)$ is the class field for the group of characters $c(m)^*$ on $c(m)$, while k_m is the class field for the group X of those characters on $c(m)$ for which $\chi(p)=1$; $X=\{\chi \in c(m)^* | \chi(p)=1\}$.

Note that X is isomorphic with $(c(m)/\langle p\rangle)^*$, where $\langle p\rangle$ is the subgroup of $c(m)$ generated by p. The discriminant-conductor formula says that

$$\operatorname{disc}(k_n) = \prod_{\chi\in X} f_\chi = \prod_{f|m} f^{a(f)}$$

where $a(f)$ is the number of elements of X with conductor f.

If $(j,p)=1$, let $e(j)$ denote the exponent of p modulo j. Then if $j|m$, the number of elements of $c(j)^*$ which are one at p is $\phi(j)/e(j)$ since the order of $c(j)$ is $\phi(j)$ and the order of $\langle p\rangle$ in $c(j)$ is $e(j)$. Hence

$$b(f) = \frac{\phi(f)}{e(f)} = \sum_{j|f} a(j),$$

since every element of X which is defined modulo f has conductor which divides f. The Möbius inversion formula gives

$$a(f) = \sum_{j|f} \mu(j) b\left(\frac{f}{j}\right) = \sum_{j|f} \mu(j) \frac{\phi(f/j)}{e(f/j)}. \tag{7}$$

The lemma will be proved by showing that $a(m)$ is sufficiently large.

Write

$$m = m_1(p^{2^{n-1}} + 1) = m_1 2m_2, \tag{8}$$

let $2^\alpha \| m$; that is, $2^\alpha | m$ and $2^{\alpha+1}$ does not divide m. It is easy to see that $(m_1, m_2) = 1$.

To use (7) to calculate $a(m)$ it is necessary to evaluate $e(m/d)$ for $d|m$ and d square free. An easy induction shows that $2^{\alpha-n-1} \| (p-1)$, so $e(2^{\alpha-\sigma}) = 2^{n-\sigma}$ when $\sigma = 0$ or 1. If $q^\beta | m_1$ then $e(q^\beta) < 2^n$ since $p^{2^{n-1}} \equiv 1 \pmod{m_1}$, while if $q^\beta \| m_2$, then (8) shows that $e(q^\beta) = 2^n$. Hence, if $d = d_1 d_2$ with d_1 odd, $d_1 | m_1$, $d_2 | 2m_2$,

$$\begin{aligned} e(m/d) &= 2^n \quad \text{if} \quad d_2 < 2m_2, \\ &= 2^{n-1} \quad \text{if} \quad d_2 = 2m_2. \end{aligned}$$

It is convenient to introduce the multiplicative function $F(\gamma) = \sum_{d|\gamma} \mu(d)\phi(\gamma/d)$, so $F(g) = g-2$, $F(g^j) = g^{j-2}(g-1)^2$ where g is prime and $j > 1$. $F(\gamma)$ is the number of characters of conductor γ, although this observation is not needed in the proof. From (7) it follows that

$$\begin{aligned} a(m) = 2^{-n} \sum_{d_1} \mu(d_1) \Bigg[\sum_{\text{odd } d_2} \mu(d_2)\phi\left(\frac{2m_1}{d_1}\right)\phi\left(\frac{m_2}{d_2}\right) \\ + \sum_{\text{even } d_2 < 2m_2} \mu(d_2)\phi\left(\frac{m_1}{d_1}\right)\phi\left(\frac{2m_2}{d_2}\right) + \mu(2m_2)2\phi\left(\frac{m_1}{d_1}\right)\Bigg] \\ = \sum_{d_1} \mu(d_1)\phi\left(\frac{m_1}{d_1}\right)\left(\sum_{\text{odd } d_2} \mu(d_2)\phi\left(\frac{m_2}{d_2}\right) - 3\mu(m_2)\right)2^{-n} \\ = 2^{\alpha-2-n} F\left(\frac{m_1}{2^{\alpha-1}}\right)(F(m_2) - 3\mu(m_2)). \end{aligned}$$

Now if γ is odd,

$$\frac{F(\gamma)}{\gamma} \gg \prod_{g|\gamma} \frac{g-2}{g} \gg \prod_{g|\gamma} \left(1 - \frac{1}{g}\right)^2 \gg (\log\log\gamma)^{-2},$$

so, since $m_1/2^{\alpha-1}$ is odd,

$$a(m) \gg 2^{\alpha-2-n} \frac{m_1}{2^{\alpha-1}(\log\log m_1)^2}\left(\frac{m_2}{(\log\log m_2)^2} - 3\right) \gg \frac{2^{-n}m}{n^4}.$$

Hence

$$\frac{\deg(k_n)}{\log(\operatorname{disc}(k_n))} \ll \frac{\phi(m)}{2^n}\frac{n^4}{2^{-n}m}\frac{1}{\log m} \ll \frac{n^4}{2^n}. \qquad \text{Q.E.D.}$$

3. Denote the primes by $2=p_1, p_2, \cdots$. Let $L_2=Q(\sqrt{7})$. If $p=p_m$, let $L_p=k_n$, where k_n is one of the fields constructed above, such that $\deg(L_p)>9^m$ and $\deg(L_p)/\log(\operatorname{disc}(L_p))<1/m$. These choices are possible by (5) and (6). Now $\Delta=0$, since each p splits in L_p by (4). Further, $n(k)\geqq\deg(L_p)>9^m$ where $p=p_m$ is the largest prime factor of k. Then since there are 2^{m-1} square free integers whose largest prime factor is p_m,

$$\sum\frac{\mu^2(h)}{n(h)}\leqq 1+\frac{1}{2}+\sum_{m=2}^{\infty}\frac{2^{m-1}}{9^m}=1+\frac{11}{14},$$

so (1) is satisfied. But (2) is not satisfied, since

$$\sum\frac{\mu(h)}{n(h)}\geqq 1-\frac{1}{2}-\sum_{m=2}^{\infty}\frac{2^{m-1}}{9^m}=\frac{3}{14}\neq 0=\Delta.$$

Bibliography

1. Larry Joel Goldstein, *Analogues of Artin's conjecture*, Bull. Amer. Math. Soc. **74** (1968), 517–519. MR **36** #6376.

2. ———, *Analogues of Artin's conjecture*, Trans. Amer. Math. Soc. **149** (1970), 431–442.

3. Helmut Hasse, *Über die Klassenzahl Abelscher Zahlkörper*, Akademie-Verlag, Berlin, 1952. MR **14**, 141.

4. Karl Prachar, *Primzahlverteilung*, Springer-Verlag, Berlin and New York, 1952. MR **19**, 393.

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48104