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NONEXISTENCE OF ASYMPTOTIC OBSERVABLES

JAMES S. HOWLAND[1]

ABSTRACT. Strong asymptotic limits of all Heisenberg observables exist only for trivial Hamiltonians.

Let H be a selfadjoint operator on a separable Hilbert space $\mathscr{H}$. Lavine [1] has introduced into scattering theory the study of the algebra of bounded operators A for which the strong asymptotic limits

$$(*) \qquad \operatorname*{s-lim}_{t\to+\infty} e^{-iHt}Ae^{iHt}$$

of the Heisenberg observables $A(t)=e^{-iHt}Ae^{iHt}$ exist. It is therefore of some interest that this algebra coincides with $\mathscr{B}(\mathscr{H})$ only in the trivial case.

THEOREM. *The limit* (*) *exists for every bounded A iff H is a constant multiple of the identity.*

PROOF. Suppose that H is not a multiple of I. If H has two distinct eigenvalues λ and μ with eigenvectors ϕ and ψ, choose $A=(\cdot, \phi)\psi$. Then $A(t)=e^{i(\lambda-\mu)t}(\cdot, \phi)\psi$ has no limit. Otherwise, H has a nontrivial continuous part, and since it suffices to construct A on a reducing subspace of H, one may assume that H is multiplication by λ on $L_2([a, b), dg)$ where $g(\lambda)$ is a continuous increasing function with $g(a)=0$ and $g(b)=1$. If every interval $[\alpha, \beta)$ on which $g(\lambda)$ is constant is deleted from $[a, b)$, the remaining set supports dg and is mapped by g in a one-one measure-preserving fashion onto $[0, 1)$ with Lebesgue measure. Under this change of variables, H becomes multiplication by the strictly increasing function $\alpha(x)=g^{-1}(x)$ on $L_2[0, 1)$. In this representation, choose $Af(x)=f(1-x)$, so that

$$A(t)f(x) = e^{-i\beta(x)t}f(1-x)$$

where $\beta(x)=\alpha(x)-\alpha(1-x)$ is strictly increasing. Since

$$\|A(t)f - A(s)f\|^2 = \int_0^1 |1 - e^{i\beta(x)(t-s)}|^2 \, |f(1-x)|^2 \, dx,$$

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which depends only on $t-s$, the strong limit can exist only if the right side vanishes for all t and s. But since $\beta(x)$ is strictly increasing, this implies that $f=0$.

REFERENCE

1. R. B. Lavine, *Scattering theory for long range potentials*, J. Functional Analysis **5** (1970), 368–382. MR **42** #5095.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22903