NONEXISTENCE OF ASYMPTOTIC OBSERVABLES

JAMES S. HOWLAND

Abstract. Strong asymptotic limits of all Heisenberg observables exist only for trivial Hamiltonians.

Let $H$ be a selfadjoint operator on a separable Hilbert space $\mathcal{H}$. Lavine [1] has introduced into scattering theory the study of the algebra of bounded operators $A$ for which the strong asymptotic limits

\[ \lim_{t \to +\infty} e^{-iHt} A e^{iHt} \]

of the Heisenberg observables $A(t) = e^{-iHt} A e^{iHt}$ exist. It is therefore of some interest that this algebra coincides with $\mathcal{B}(\mathcal{H})$ only in the trivial case.

Theorem. The limit (*) exists for every bounded $A$ iff $H$ is a constant multiple of the identity.

Proof. Suppose that $H$ is not a multiple of $I$. If $H$ has two distinct eigenvalues $\lambda$ and $\mu$ with eigenvectors $\phi$ and $\psi$, choose $A = (\cdot, \phi)\psi$. Then $A(t) = e^{i(\lambda - \mu)t}(\cdot, \phi)\psi$ has no limit. Otherwise, $H$ has a nontrivial continuous part, and since it suffices to construct $A$ on a reducing subspace of $H$, one may assume that $H$ is multiplication by $\lambda$ on $L^2([a, b], dg)$ where $g(\lambda)$ is a continuous increasing function with $g(a) = 0$ and $g(b) = 1$. If every interval $[a, \beta)$ on which $g(\lambda)$ is constant is deleted from $[a, b)$, the remaining set supports $dg$ and is mapped by $g$ in a one-one measure-preserving fashion onto $[0, 1)$ with Lebesgue measure. Under this change of variables, $H$ becomes multiplication by the strictly increasing function $\alpha(x) = g^{-1}(x)$ on $L^2[0, 1)$. In this representation, choose $A f(x) = f(1 - x)$, so that

\[ A(t)f(x) = e^{-i\beta(x)t} f(1 - x) \]

where $\beta(x) = \alpha(x) - \alpha(1 - x)$ is strictly increasing. Since

\[ \| A(t)f - A(s)f \|^2 = \int_0^1 |1 - e^{i\beta(x)(t-s)}|^2 |f(1 - x)|^2 \, dx, \]

Received by the editors November 8, 1971.

AMS 1970 subject classifications. Primary 47A40; Secondary 81A48.

1 Supported by ARO Grant DA-ARO-D-31-124-G1005.
which depends only on \( t - s \), the strong limit can exist only if the right side vanishes for all \( t \) and \( s \). But since \( \beta(x) \) is strictly increasing, this implies that \( f = 0 \).

**Reference**


**Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903**