NONEXISTENCE OF ASYMPTOTIC OBSERVABLES

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Abstract. Strong asymptotic limits of all Heisenberg observables exist only for trivial Hamiltonians.

Let $H$ be a selfadjoint operator on a separable Hilbert space $\mathcal{H}$. Lavine [1] has introduced into scattering theory the study of the algebra of bounded operators $A$ for which the strong asymptotic limits

\[ \text{s-lim}_{t \to \pm \infty} e^{-iHt} A e^{iHt} \]

of the Heisenberg observables $A(t) = e^{-iHt} A e^{iHt}$ exist. It is therefore of some interest that this algebra coincides with $B(\mathcal{H})$ only in the trivial case.

Theorem. The limit (*) exists for every bounded $A$ iff $H$ is a constant multiple of the identity.

Proof. Suppose that $H$ is not a multiple of $I$. If $H$ has two distinct eigenvalues $\lambda$ and $\mu$ with eigenvectors $\phi$ and $\psi$, choose $A = (\cdot, \phi)\psi$. Then $A(t) = e^{i(\lambda - \mu)t}(\cdot, \phi)\psi$ has no limit. Otherwise, $H$ has a nontrivial continuous part, and since it suffices to construct $A$ on a reducing subspace of $H$, one may assume that $H$ is multiplication by $\lambda$ on $L_2([a, b), dg)$ where $g(\lambda)$ is a continuous increasing function with $g(a) = 0$ and $g(b) = 1$. If every interval $[\alpha, \beta)$ on which $g(\lambda)$ is constant is deleted from $[a, b)$, the remaining set supports $dg$ and is mapped by $g$ in a one-one measure-preserving fashion onto $[0, 1)$ with Lebesgue measure. Under this change of variables, $H$ becomes multiplication by the strictly increasing function $x(x) = g^{-1}(x)$ on $L_2[0, 1)$. In this representation, choose $A f(x) = f(1 - x)$, so that

\[ A(t)f(x) = e^{-i\beta(x)t}f(1 - x) \]

where $\beta(x) = x(x) - x(1 - x)$ is strictly increasing. Since

\[ \|A(t)f - A(s)f\|^2 = \int_0^1 |1 - e^{i\beta(x)(t-s)}|^2 |f(1 - x)|^2 \, dx, \]

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which depends only on $t-s$, the strong limit can exist only if the right side
vanishes for all $t$ and $s$. But since $\beta(x)$ is strictly increasing, this implies
that $f=0$.

**REFERENCE**