CONTRACTIBLE HILBERT CUBE MANIFOLDS

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Abstract. In this note we give an example of a contractible Hilbert cube manifold which cannot be embedded as an open subset of the Hilbert cube $Q$ so that its complement (in $Q$) lies in a face of the boundary of $Q$. This example provides a negative answer to a question raised by the author.

1. Introduction. Let the Hilbert cube $Q$ be represented by $\prod_{i=1}^{\infty} I_i$, where each $I_i$ is the closed interval $[0, 1]$, and let $W=\{(x_i) \in Q | x_1 = 1\}$. By a Hilbert cube manifold (or $Q$-manifold) we mean a separable metric space for which each point has a closed neighborhood homeomorphic to $Q$. In this paper we will only be concerned with noncompact contractible $Q$-manifolds, since all compact contractible $Q$-manifolds are homeomorphic to $Q$ [3]. If $A \subset W$ is a closed set, then it is clear that $Q \setminus A$ is a contractible $Q$-manifold. In general define a $Q$-manifold to be of type $Q$ provided that it is homeomorphic to $Q \setminus A$, for some closed $A \subset W$. We remark that any closed set $A \subset W$ has Property Z in $Q$ (in the sense of Anderson [1]), but this concept will not be used in this paper. In fact we use no infinite-dimensional topology other than some elementary topological properties of $Q$.

The $Q$-manifolds of type $Q$ form a particularly distinguished collection because of the following representation and characterization theorems.

Theorem 1 ([3]). If $X$ is a $Q$-manifold of type $Q$, then there exists a countable locally-finite simplicial complex $K$ such that $X$ is homeomorphic to $|K| \times Q$.

Theorem 2 ([5]). If $X$ and $Y$ are $Q$-manifolds of type $Q$ which have the same proper homotopy type, then $X$ and $Y$ are homeomorphic.

(See §2 for definitions concerning the proper category.)

Attempting to extend these results to the class of all contractible $Q$-manifolds one is naturally led to the following question (see Question 4 of [4] and QM5 of [1]):

Question. Is every contractible $Q$-manifold of type $Q$?
It is obvious that an affirmative answer to this question would extend Theorem 1 and Theorem 2 to the class of all contractible Q-manifolds.

The objective of this note is to answer the above question negatively by giving an example of a contractible Q-manifold which is not of type Q. If M is Whitehead's well-known example of an open contractible 3-manifold which is not homeomorphic to $E^3$ [7], then we prove that $M \times Q$ is a contractible Q-manifold which is not of type Q. To be more precise (and in the language of §2) we prove that $M \times Q$ is not proper homotopically dominated by any Q-manifold of the form $Q \setminus A$, where $A \subset W$ is closed. We also remark that this example sharpens the following result of [3] concerning decompositions of contractible Q-manifolds into open sets of type Q: Any contractible Q-manifold can be written as the union of two dense open sets, each one of which is of type Q.

The question of extending Theorem 1 and Theorem 2 to the class of all contractible Q-manifolds remains open. The example given in this paper merely rules out one method of attack.

2. Proper maps. We give here some basic definitions concerning proper maps, and in Lemma 2.1 we establish a result which will be used in §3.

For spaces $X$ and $Y$ we say that a map (i.e. a continuous function) $f: X \to Y$ is proper provided that $f^{-1}(C)$ is compact, for all compact sets $C \subset Y$. Two proper maps $f, g: X \to Y$ are said to be proper homotopic (written $f \simeq_p g$) provided that there exists a proper map $F: X \times [0, 1] \to Y$ which satisfies $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in X$. (We reserve the symbol $\simeq$ for the usual notion of homotopic maps.) We say that spaces $X$ and $Y$ have the same proper homotopy type provided that there exist proper maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f \simeq_p \text{id}_X$ and $f \circ g \simeq_p \text{id}_Y$ (where $\text{id}_X$ and $\text{id}_Y$ denote the identity maps of $X$ and $Y$, respectively). As an analogue of the concept of homotopy domination we say that a space $X$ proper homotopically dominates a space $Y$ provided that there exist proper maps $f: X \to Y$ and $g: Y \to X$ such that $f \circ g \simeq_p \text{id}_X$.

The following result will be needed in §3. It pinpoints a property of locally compact metric spaces which are proper homotopically dominated by $Q \setminus A$, for some closed $A \subset W$.

**Lemma 2.1.** Let $X$ be a locally compact metric space which is proper homotopically dominated by $Q \setminus A$, for some closed $A \subset W$. Then for each compact set $K_1 \subset X$ there exists a compact set $K_2 \subset X$ containing $K_1$ and there exists a map $F: (X \setminus K_2) \times [0, 1] \to X \setminus K_1$ such that $F(x, 0) = x$ and $F(x, 1) \in K_2 \setminus K_1$, for all $x \in X \setminus K_2$. 
Proof. We will first establish the above property for the space $Q \setminus A$. Thus let $K'_1 \subseteq Q \setminus A$ be a compact set; we want to find a compact set $K'_2 \subseteq Q \setminus A$ which contains $K'_1$ and a map $F' : (Q \setminus (A \cup K'_2)) \times [0, 1] \to Q \setminus (A \cup K'_2)$ such that $F'(x, 0) = x$ and $F'(x, 1) \in K'_2 \setminus K'_1$, for all $x \in Q \setminus (A \cup K'_2)$.

Note that $Q \setminus K'_1$ is an open set containing $A$. Choose $V$ to be a relatively open subset of $W$ such that $A \subseteq V \subseteq Cl'_W(V) \subseteq Q \setminus K'_1$, where $Cl'_W(V)$ is the closure of $V$ in $W$. Let $Bd'_W(V)$ be the topological boundary of $V$ in $W$ and let $p : W \to \bigsqcup_{i=2}^n I$, be projection. Clearly there exists a map $r : Cl'_W(V) \to [0, 1]$ such that $r(V) = (0, 1)$, $r(Bd'_W(V)) = \{1\}$, and $[r(x), 1] \times \{p(x)\} \subseteq Q \setminus K'_1$, for all $x \in Cl'_W(V)$. Let

$$U = \bigcup \{ (r(x), 1) \times \{p(x)\} \mid x \in V \},$$

which is an open subset of $Q$ containing $A$. Then we put $K'_2 = Q \setminus U$.

Note that $U \setminus A = Q \setminus (A \cup K'_2)$ and define $F' : (U \setminus A) \times [0, 1] \to Q \setminus (A \cup K'_2)$ by setting $F'((s, p(x)), (t)) = (((1 - t)s + tr(x), p(x)) \in [r(x), 1] \times \{p(x)\}$, for all $x \in V$, $(s, p(x)) \in (r(x), 1) \times \{p(x)\}) \setminus A$, and $t \in [0, 1]$. It is clear that $F'$ fulfills our requirements.

Passing to the general case let $f : X \to Q \setminus A$ and $g : Q \setminus A \to X$ be proper maps such that $g \circ f \cong id_X$ and let $K_1 \subseteq X$ be a given compact set. Since $g$ is proper there exists a compact set $K'_1 \subseteq Q \setminus A$ such that $g(Q \setminus (A \cup K'_1)) \cap K_1 = \emptyset$. Using the special case treated above there exist a compact set $K'_2 \subseteq Q \setminus A$ containing $K'_1$ and a map $F' : (Q \setminus (A \cup K'_2)) \times [0, 1] \to Q \setminus (A \cup K'_2)$ such that $F'(x, 0) = x$ and $F'(x, 1) \in K'_2 \setminus K'_1$, for all $x \in Q \setminus (A \cup K'_2)$.

Since $f$ is proper and $g \circ f \cong id_X$ we can find a compact set $K'_2 \subseteq X$ containing $K_1$ such that $f(X \setminus K'_2) \subseteq Q \setminus (A \cup K'_2)$ and we can find a map $G : (X \setminus K'_2) \times [0, 1] \to X \setminus K_1$ such that $G(x, 0) = x$ and $G(x, 1) = g \circ f(x)$, for all $x \in X \setminus K'_2$. Define $F^* : (X \setminus K'_2) \times [0, 1] \to X \setminus K_1$ by setting

$$F^*(x, t) = G(x, 2t)$$

$$= g \circ F'(f(x), 2t - 1)$$

for all $(x, t) \in (X \setminus K'_2) \times [0, 1]$. Note that $F^*(x, 0) = x$ and $F^*(x, 1) = g \circ F'(f(x), 1) \in g(K'_2)$, for all $x \in X \setminus K'_2$. If $K_2$ is a compact set in $X$ containing $g(K'_2) \cup K'_2$ and $F = F^*\mid (X \setminus K'_2) \times [0, 1]$, then we have $F(x, 1) = g \circ F'(f(x), 1) \in g(K'_2) \subseteq K_2$, for all $x \in X \setminus K'_2$. Thus we have constructed $K_2$ and $F$ to fulfill our requirements.

3. The example. We will first need to describe Whitehead's example of an open contractible 3-manifold which is topologically different from $E^3$ [7]. We use the description of Bing [2]. Consider a solid torus $T_1$ embedded in a solid torus $T_2$ as shown in the figure below (where $T_2$ is a subset of $E^3$). Since $T_1$ is tame in $E^3$ there exists a homeomorphism $h$ of $E^3$ onto itself such that $h(T_1) = T_2$. Let $M = T_1 \cup h(T_1) \cup h^2(T_1) \cup \cdots$, which
is Whitehead's open contractible 3-manifold. Let \( T_{n+1} \) be the torus \( h^n(T_1) \), for all \( n \geq 2 \).

In [7] Whitehead proved that \( M \) is not PL homeomorphic to \( E^3 \). In [6] Newman and Whitehead examined the fundamental groups of various subsets of \( M \) and used this information to prove that \( M \) is not homeomorphic to \( E^3 \). As a consequence of the work in [6] we have the following:

1. \( \pi_1(M \setminus T_1) \) is not finitely generated (f.g.).
2. \( \pi_1(T_n \setminus T_1) \) is f.g., for all \( n \geq 1 \).

Since \( M \) is a manifold it is obvious that \( M \times Q \) is a \( Q \)-manifold, therefore a contractible \( Q \)-manifold (since \( M \) is contractible). We now proceed to prove that \( M \times Q \) is not of type \( Q \). We do this by showing that the assumption that \( M \times Q \) is of type \( Q \) leads to a contradiction. The assumption that \( M \times Q \) is of type \( Q \) implies that \( M \times Q \) is proper homotopically dominated by \( Q \cup A \), for some closed \( A \subset W \). Since \( Q \) is compact and contractible, it easily follows that \( M \) and \( M \times Q \) have the same proper homotopy type. Thus \( M \) is proper homotopically dominated by \( Q \cup A \).

Using Lemma 2.1 there exists a compact set \( C \subset M \) containing \( T_1 \) and a map \( F : (M \setminus C) \times [0, 1] \rightarrow M \setminus T_1 \) such that \( F(x, 0) = x \) and \( F(x, 1) \in C \setminus T_1 \), for all \( x \in M \setminus C \).

Now choose \( n > 1 \) large enough so that \( C \subset T_n \). It is obvious that \( \partial T_n = \text{Bd}_M(T_n) \) is collared in \( T_n \setminus C \), i.e. there exists an embedding \( f: \partial T_n \times [0, 1] \rightarrow T_n \setminus C \) such that \( f(\partial T_n \times [0, 1]) \) is open in \( T_n \) and \( f(x, 0) = x \), for all \( x \in \partial T_n \). We now define a map \( G : (M \setminus T_1) \times [0, 1] \rightarrow M \setminus T_1 \). To define \( G \) on \((M \setminus f(\partial T_n \times [0, 1])) \times [0, 1]\) we put

\[
G(x, t) = x \quad \text{for } x \in T_n \setminus f(\partial T_n \times [0, 1]), \ t \in [0, 1],
\]

\[
= F(x, t) \quad \text{for } x \in M \setminus T_n, \ t \in [0, 1].
\]

To define \( G \) on \( f(\partial T_n \times [0, 1]) \times [0, 1] \) choose any \( x \in \partial T_n \) and put \( G(f(x, s), t) = F(f(x, s), (1-s)t) \), for all \( x, \ t \in [0, 1] \). It is clear that \( G \) satisfies \( G(x, 0) = x \), for all \( x \in M \setminus T_1 \). Since \( F(\partial T_n \times [0, 1]) \times [0, 1] \) is
a compact subset of $M \setminus T_1$, there exists an integer $m \geq n$ such that 
$F(f(\partial T_m \times [0, 1]) \times [0, 1]) \subseteq T_m \setminus T_1$. It then follows that $G(x, 1) \in T_m \setminus T_1$, for all $x \in M \setminus T_1$.

Choose any $x_0 \in C/T_1$ and let $\sigma: I \to M \setminus T_1$ be a loop at $x_0$, i.e. $\sigma: I \to M \setminus T_1$ is a map and $\sigma(0) = \sigma(1) = x_0$. For each $t \in [0, 1]$ define a loop $\sigma_t: I \to M \setminus T_1$ by $\sigma_t(s) = G(\sigma(s), t)$, for all $s \in [0, 1]$. Then each $\sigma_t$ is a loop at $x_0$, $\sigma_0 = \sigma$, and $\sigma_t$ is a loop in $T_m \setminus T_1$. This proves that each loop in $M \setminus T_1$ at $x_0$ can be continuously deformed to a loop in $T_m \setminus T_1$ at $x_0$ (with the point $x_0$ being kept fixed at each level of the deformation). Restating this it means that if $i: T_m \setminus T_1 \to M \setminus T_1$ is the inclusion map, then

$$i_*: \pi_1(T_m \setminus T_1, x_0) \to \pi_1(M \setminus T_1, x_0)$$

is a surjection. But this contradicts the facts that $\pi_1(T_m \setminus T_1)$ is f.g. and $\pi_1(M \setminus T_1)$ is not f.g.

**References**

2. R. H. Bing, Necessary and sufficient conditions that a $2$-manifold be $S^2$, Ann. of Math. (2) 68 (1958), 17-37. MR 20#1973
7. J. H. C. Whitehead, A certain open manifold whose group is unity, Quart. J. Math. 6 (1935), 268-279