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CONTRACTIBLE HILBERT CUBE MANIFOLDS

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ABSTRACT. In this note we give an example of a contractible Hilbert cube manifold which cannot be embedded as an open subset of the Hilbert cube Q so that its complement (in Q) lies in a face of the boundary of Q. This example provides a negative answer to a question raised by the author.

1. **Introduction.** Let the Hilbert cube Q be represented by $\prod_{i=1}^{\infty} I_i$, where each I_i is the closed interval [0, 1], and let $W=\{(x_i)\in Q \mid x_1=1\}$. By a *Hilbert cube manifold* (or Q-manifold) we mean a separable metric space for which each point has a closed neighborhood homeomorphic to Q. In this paper we will only be concerned with noncompact contractible Q-manifolds, since all compact contractible Q-manifolds are homeomorphic to Q [3]. If $A\subset W$ is a closed set, then it is clear that $Q\backslash A$ is a contractible Q-manifold. In general define a Q-manifold to be of *type Q* provided that it is homeomorphic to $Q\backslash A$, for some closed $A\subset W$. We remark that any closed set $A\subset W$ has Property Z in Q (in the sense of Anderson [1]), but this concept will not be used in this paper. In fact we use no infinite-dimensional topology other than some elementary topological properties of Q.

The Q-manifolds of type Q form a particularly distinguished collection because of the following representation and characterization theorems.

THEOREM 1 ([3]). *If X is a Q-manifold of type Q, then there exists a countable locally-finite simplicial complex K such that X is homeomorphic to $|K|\times Q$.*

THEOREM 2 ([5]). *If X and Y are Q-manifolds of type Q which have the same proper homotopy type, then X and Y are homeomorphic.*

(See §2 for definitions concerning the *proper category*.)

Attempting to extend these results to the class of all contractible Q-manifolds one is naturally led to the following question (see Question 4 of [4] and $QM5$ of [1]):

Question. Is every contractible Q-manifold of type Q?

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It is obvious that an affirmative answer to this question would extend Theorem 1 and Theorem 2 to the class of all contractible Q-manifolds.

The objective of this note is to answer the above question negatively by giving an example of a contractible Q-manifold which is not of type Q. If M is Whitehead's well-known example of an open contractible 3-manifold which is not homeomorphic to E^3 [7], then we prove that $M \times Q$ is a contractible Q-manifold which is not of type Q. To be more precise (and in the language of §2) we prove that $M \times Q$ is not *proper homotopically dominated* by any Q-manifold of the form $Q \backslash A$, where $A \subset W$ is closed. We also remark that this example sharpens the following result of [3] concerning decompositions of contractible Q-manifolds into open sets of type Q: *Any contractible Q-manifold can be written as the union of two dense open sets, each one of which is of type Q.*

The question of extending Theorem 1 and Theorem 2 to the class of all contractible Q-manifolds remains open. The example given in this paper merely rules out one method of attack.

2. **Proper maps.** We give here some basic definitions concerning proper maps, and in Lemma 2.1 we establish a result which will be used in §3.

For spaces X and Y we say that a map (i.e. a continuous function) $f: X \to Y$ is *proper* provided that $f^{-1}(C)$ is compact, for all compact sets $C \subset Y$. Two proper maps $f, g: X \to Y$ are said to be *proper homotopic* (written $f \simeq_p g$) provided that there exists a proper map $F: X \times [0, 1] \to Y$ which satisfies $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in X$. (We reserve the symbol $\simeq$ for the usual notion of homotopic maps.) We say that spaces X and Y have the same *proper homotopy type* provided that there exist proper maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f \simeq_p \mathrm{id}_X$ and $f \circ g \simeq_p \mathrm{id}_Y$ (where id_X and id_Y denote the identity maps of X and Y, respectively). As an analogue of the concept of homotopy domination we say that a space X *proper homotopically dominates* a space Y provided that there exist proper maps $f: X \to Y$ and $g: Y \to X$ such that $f \circ g \simeq_p \mathrm{id}_Y$.

The following result will be needed in §3. It pinpoints a property of locally compact metric spaces which are proper homotopically dominated by $Q \backslash A$, for some closed $A \subset W$.

LEMMA 2.1. *Let X be a locally compact metric space which is proper homotopically dominated by $Q \backslash A$, for some closed $A \subset W$. Then for each compact set $K_1 \subset X$ there exists a compact set $K_2 \subset X$ containing K_1 and there exists a map*

$$F: (X \backslash K_2) \times [0, 1] \to X \backslash K_1$$

such that $F(x, 0) = x$ and $F(x, 1) \in K_2 \backslash K_1$, for all $x \in X \backslash K_2$.

PROOF. We will first establish the above property for the space $Q\backslash A$. Thus let $K_1'\subset Q\backslash A$ be a compact set; we want to find a compact set $K_2'\subset Q\backslash A$ which contains K_1' and a map $F':(Q\backslash(A\cup K_2'))\times[0,1]\to Q\backslash(A\cup K_1')$ such that $F'(x,0)=x$ and $F'(x,1)\in K_2'\backslash K_1'$, for all $x\in Q\backslash(A\cup K_2')$.

Note that $Q\backslash K_1'$ is an open set containing A. Choose V to be a relatively open subset of W such that $A\subset V\subset \mathrm{Cl}_W(V)\subset Q\backslash K_1'$, where $\mathrm{Cl}_W(V)$ is the closure of V in W. Let $\mathrm{Bd}_W(V)$ be the topological boundary of V in W and let $p: W\to\prod_{i=2}^{\infty} I_i$ be projection. Clearly there exists a map $r:\mathrm{Cl}_W(V)\to[0,1]$ such that $r(V)\subset(0,1)$, $r(\mathrm{Bd}_W(V))=\{1\}$, and $[r(x),1]\times\{p(x)\}\subset Q\backslash K_1'$, for all $x\in\mathrm{Cl}_W(V)$. Let

$$U=\bigcup\{(r(x),1]\times\{p(x)\}\mid x\in V\},$$

which is an open subset of Q containing A. Then we put $K_2'=Q\backslash U$.

Note that $U\backslash A=Q\backslash(A\cup K_2')$ and define $F':(U\backslash A)\times[0,1]\to Q\backslash(A\cup K_1')$ by setting $F'((s,p(x)),t)=((1-t)s+tr(x),p(x))\in[r(x),1]\times\{p(x)\}$, for all $x\in V$, $(s,p(x))\in((r(x),1]\times\{p(x)\})\backslash A$, and $t\in[0,1]$. It is clear that F' fulfills our requirements.

Passing to the general case let $f: X\to Q\backslash A$ and $g: Q\backslash A\to X$ be proper maps such that $g\circ f\simeq_p \mathrm{id}_X$ and let $K_1\subset X$ be a given compact set. Since g is proper there exists a compact set $K_1'\subset Q\backslash A$ such that $g(Q\backslash(A\cup K_1'))\cap K_1=\varnothing$. Using the special case treated above there exist a compact set $K_2'\subset Q\backslash A$ containing K_1' and a map $F':(Q\backslash(A\cup K_2'))\times[0,1]\to Q\backslash(A\cup K_1')$ such that $F'(x,0)=x$ and $F'(x,1)\in K_2'\backslash K_1'$, for all $x\in Q\backslash(A\cup K_2')$.

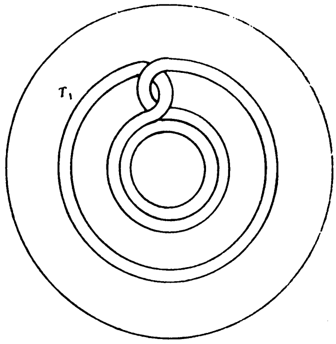
Since f is proper and $g\circ f\simeq_p \mathrm{id}_X$ we can find a compact set $K_2^*\subset X$ containing K_1 such that $f(X\backslash K_2^*)\subset Q\backslash(A\cup K_2')$ and we can find a map $G:(X\backslash K_2^*)\times[0,1]\to X\backslash K_1$ such that $G(x,0)=x$ and $G(x,1)=g\circ f(x)$, for all $x\in X\backslash K_2^*$. Define $F^*:(X\backslash K_2^*)\times[0,1]\to X\backslash K_1$ by setting

$$\begin{aligned} F^*(x,t) &= G(x,2t) && 0\leqq t\leqq \tfrac{1}{2},\\ &= g\circ F'(f(x),2t-1) && \tfrac{1}{2}\leqq t\leqq 1, \end{aligned}$$

for all $(x,t)\in(X\backslash K_2^*)\times[0,1]$. Note that $F^*(x,0)=x$ and $F^*(x,1)=g\circ F'(f(x),1)\in g(K_2')$, for all $x\in X\backslash K_2^*$. If K_2 is a compact set in X containing $g(K_2')\cup K_2^*$ and $F=F^*|(X\backslash K_2)\times[0,1]$, then we have $F(x,1)=g\circ F'(f(x),1)\in g(K_2')\subset K_2$, for all $x\in X\backslash K_2$. Thus we have constructed K_2 and F to fulfill our requirements.

3. **The example.** We will first need to describe Whitehead's example of an open contractible 3-manifold which is topologically different from E^3 [7]. We use the description of Bing [2]. Consider a solid torus T_1 embedded in a solid torus T_2 as shown in the figure below (where T_2 is a subset of E^3). Since T_1 is tame in E^3 there exists a homeomorphism h of E^3 onto itself such that $h(T_1)=T_2$. Let $M=T_1\cup h(T_1)\cup h^2(T_1)\cup\cdots$, which

is Whitehead's open contractible 3-manifold. Let T_{n+1} be the torus $h^n(T_1)$, for all $n \geqq 2$.



In [7] Whitehead proved that M is not PL homeomorphic to E^3. In [6] Newman and Whitehead examined the fundamental groups of various subsets of M and used this information to prove that M is not homeomorphic to E^3. As a consequence of the work in [6] we have the following:

1. *$\pi_1(M \backslash T_1)$ is not finitely generated (f.g.).*
2. *$\pi_1(T_n \backslash T_1)$ is f.g., for all $n > 1$.*

Since M is a manifold it is obvious that $M \times Q$ is a Q-manifold, therefore a contractible Q-manifold (since M is contractible). We now proceed to prove that $M \times Q$ is not of type Q. We do this by showing that the assumption that $M \times Q$ is of type Q leads to a contradiction. The assumption that $M \times Q$ is of type Q implies that $M \times Q$ is proper homotopically dominated by $Q \backslash A$, for some closed $A \subset W$. Since Q is compact and contractible, it easily follows that M and $M \times Q$ have the same proper homotopy type. Thus M is proper homotopically dominated by $Q \backslash A$. Using Lemma 2.1 there exists a compact set $C \subset M$ containing T_1 and a map $F:(M \backslash C) \times [0, 1] \to M \backslash T_1$ such that $F(x, 0) = x$ and $F(x, 1) \in C \backslash T_1$, for all $x \in M \backslash C$.

Now choose $n > 1$ large enough so that $C \subset T_n$. It is obvious that $\partial T_n = \mathrm{Bd}_M(T_n)$ is *collared* in $T_n \backslash C$, i.e. there exists an embedding $f: \partial T_n \times [0, 1] \to T_n \backslash C$ such that $f(\partial T_n \times [0, 1))$ is open in T_n and $f(x, 0) = x$, for all $x \in \partial T_n$. We now define a map $G:(M \backslash T_1) \times [0, 1] \to M \backslash T_1$. To define G on $(M \backslash f(\partial T_n \times (0, 1))) \times [0, 1]$ we put

$$\begin{aligned} G(x, t) &= x && \text{for } x \in T_n \backslash f(\partial T_n \times [0, 1)),\ t \in [0, 1], \\ &= F(x, t) && \text{for } x \in M \backslash T_n^\circ,\ t \in [0, 1]. \end{aligned}$$

To define G on $f(\partial T_n \times [0, 1]) \times [0, 1]$ choose any $x \in \partial T_n$ and put $G(f(x, s), t) = F(f(x, s), (1-s)t)$, for all s, $t \in [0, 1]$. It is clear that G satisfies $G(x, 0) = x$, for all $x \in M \backslash T_1$. Since $F(f(\partial T_n \times [0, 1]) \times [0, 1])$ is a compact subset of $M\setminus T_1$ there exists an integer $m \geqq n$ such that $F(f(\partial T_n \times [0, 1]) \times [0, 1]) \subset T_m \setminus T_1$. It then follows that $G(x, 1) \in T_m \setminus T_1$, for all $x \in M\setminus T_1$.

Choose any $x_0 \in C/T_1$ and let $\sigma: I \to M\setminus T_1$ be a loop at x_0, i.e. $\sigma: I \to M\setminus T_1$ is a map and $\sigma(0)=\sigma(1)=x_0$. For each $t \in [0, 1]$ define a loop $\sigma_t: I \to M\setminus T_1$ by $\sigma_t(s)=G(\sigma(s), t)$, for all $s \in [0, 1]$. Then each σ_t is a loop at x_0, $\sigma_0=\sigma$, and σ_1 is a loop in $T_m\setminus T_1$. This proves that each loop in $M\setminus T_1$ at x_0 can be continuously deformed to a loop in $T_m\setminus T_1$ at x_0 (with the point x_0 being kept fixed at each level of the deformation). Restating this it means that if $i: T_m\setminus T_1 \to M\setminus T_1$ is the inclusion map, then

$$i_*: \pi_1(T_m\setminus T_1, x_0) \to \pi_1(M\setminus T_1, x_0)$$

is a surjection. But this contradicts the facts that $\pi_1(T_m\setminus T_1)$ is f.g. and $\pi_1(M\setminus T_1)$ is not f.g.

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