PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, No. 1, September 1972

AN APPLICATION OF THE ALGEBRA OF DIFFERENTIALS OF INFINITE RANK

WILLIAM C. BROWN

ABSTRACT. Let k denote an arbitrary field and let R be an affine local domain over k. Let $(\Omega_k(R), \delta_k^R)$ be the universal algebra of k-higher differentials over R. Let K be the quotient field of R and L the residue class field of R. If K is a separable extension of k and L is a separable algebraic extension of k, then it is shown that R is a regular local ring if and only if $\Omega_k(R)$ is a free R-algebra. If both K and L are separable extensions of k and R has a separating residue class field, then R is a regular local ring if and only if $\Omega_k(R)$ is a free R-algebra.

Introduction. Let k and A denote commutative associative rings with identities. We assume throughout this paper that A is a k-algebra. Thus we have a ring homomorphism $\theta:k\to A$ which sends the identity of k to the identity of A. If x is an element of k and b an element of A, we shall denote $\theta(x)b$ by just xb. By an A-algebra V, we shall mean a commutative and associative ring V which is a unitary A-module, and for all v_1, v_2 in V and a in A we have $a(v_1v_2)=(av_1)v_2=v_1(av_2)$.

A k-higher derivation $\delta=\{\delta_i\}$ of A into V is an infinite sequence $\delta_1, \delta_2, \delta_3, \cdots$ of maps $\delta_i:A\to V$ such that

(a) each δ_i is an element of $\mathrm{Hom}_k(A, V)$;

(b) for all a and b in A and $i\geqq 1$, we have

$$\delta_i(ab) = a\delta_i(b) + \delta_1(a)\delta_{i-1}(b) + \cdots + \delta_{i-1}(a)\delta_1(b) + \delta_i(a)b.$$

We shall abbreviate this last line by writing

$$\delta_i(ab) = \sum_{j+k=i} \delta_j(a)\delta_k(b).$$

In [2], the author and W. E. Kuan used this notion of k-higher derivations to obtain some new results on analytic products of a variety V along a subvariety W. In this note, we shall show that higher derivations can be used to study simple points of algebraic varieties.

Received by the editors November 9, 1971.

AMS 1970 *subject classifications*. Primary 13H05.

Key words and phrases. k-higher derivations, algebra of k-higher differentials, separating residue class field, separating representatives.

In [1][1], the present author introduced the universal algebra $\Omega_k(A)$ of higher differentials on A and explored the functorial properties of $\Omega_k(\ )$. $\Omega_k(A)$ is an A-algebra (without identity) which has the following properties:

(a) There exists a canonical k-higher derivation $\delta_k^A=\{\delta_{ki}^A\}$ of A into $\Omega_k(A)$.

(b) $\Omega_k(A)$ is generated as an A-algebra by the set $\{\delta_{ki}^A(a)|a \in A, i \geqq 1\}$.

(c) $(\Omega_k(A), \delta_k^A)$ has the following universal mapping property: If V is any A-algebra and $\lambda=\{\lambda_i\}$ a k-higher derivation of A into V, then there exists a unique A-algebra homomorphism $\psi:\Omega_k(A)\rightarrow V$ such that for all $i\geqq 1$ we have $\lambda_i=\psi\delta_{ki}^A$.

The construction of $\Omega_k(A)$ is briefly as follows: Let I be the kernel of the multiplication map $P:A \otimes_k A\rightarrow A$ given by $P(\sum x_j\otimes y_j)=\sum x_jy_j$. Let I' be the direct sum of the A-modules I/I^{i+1} for $i\geqq 1$. Thus

$$I' = \sum_{i\geqq 1} \oplus I/I^{i+1}.$$

We now form the symmetric algebra $S(I')$ of I' over A but without terms of degree zero. Thus

$$S(I') = I' \oplus \left\{\frac{I' \otimes_A I'}{B_2}\right\} \oplus \left\{\frac{I' \otimes_A I' \otimes_A I'}{B_3}\right\} \oplus \cdots$$

where B_n is the A-submodule of $I' \otimes_A I' \otimes_A \cdots \otimes_A I'$ (n factors) generated by all elements of the form $s_1 \otimes_A \cdots \otimes_A s_n - s_{\mathfrak{G}(1)} \otimes_A \cdots \otimes_A s_{\mathfrak{G}(n)}$. Here $s_1, \cdots, s_n$ are elements of I' and $\mathfrak{G}$ is any permutation of $\{1, \cdots, n\}$. The mapping $\delta^i:A\rightarrow I/I^{i+1}$ defined by

$$\delta^i(a) = (1 \otimes_k a - a \otimes_k 1) + I^{i+1}$$

is an ith order k-derivation of A into I/I^{i+1} [5]. Thus each δ^i induces an ith order k-derivation of A into $S(I')$. Let J denote the ideal in $S(I')$ which is generated by all elements of the form

$$\delta^i(ab) - \sum_{j+k=i} \delta^j(a)\delta^k(b), \qquad a, b \text{ in } A,\ i \geqq 1.$$

Then $S(I')/J=\Omega_k(A)$ and δ_{ki}^A is the induced map from δ^i. One can now show that $(\Omega_k(A), \delta_k^A)$ satisfy conditions a, b and c [**1**, Theorem 1].

In this paper, we shall show that under suitable conditions the simplicity of a point on a variety defined over a field k can be determined by studying $\Omega_k(R)$, R the local ring at the point in question.

---

[1] It has recently come to my attention that P. Ribenboim in *Higher derivations of rings*. I (Rev. Roumaine Math. Pures Appl., Tome XVI, Bucarest, 1971, pp. 77–110) has also constructed a universal object using different techniques than those in [**1**].

Main results. Let $(\Omega_k^1(A), \delta_{A/k}^1)$ denote the universal object for k-derivations of rank one (ordinary derivations) on A ([**4**] or [**5**]). We need the following important lemma:

LEMMA. *$\Omega_k^1(A)$ is an A-module direct summand of $\Omega_k(A)$.*

PROOF. It is well known [**5**, p. 15] that $\Omega_k^1(A)=I/I^2$ and $\delta_{A/k}^1(a)=(1\otimes_k a - a\otimes_k 1)+I^2$. We note that I/I^2 is an A-algebra in which the product of any two elements is zero. This allows us to extend $\delta_{A/k}^1$ to a k-higher derivation $\delta=\{\delta_i\}:A\rightarrow\Omega_k^1(A)$ in the following way: For each $i\geqq 1$, define $\delta_i=\delta_{A/k}^1$. Then each δ_i is a k-linear mapping of A into $\Omega_k^1(A)$ and, for all $i\geqq 1$,

$$\delta_i(ab) = \sum_{j+i=k} \delta_j(a)\delta_k(b).$$

Now from the universal mapping property of $(\Omega_k(A), \delta_k^A)$, there exists a unique A-algebra homomorphism $\psi_1:\Omega_k(A)\rightarrow\Omega_k^1(A)$ such that for all $i\geqq 1$, $\psi_1\delta_{ki}^A=\delta_i=\delta_{A/k}^1$. From the universal mapping property of $(\Omega_k^1(A), \delta_{A/k}^1)$, there exists a unique A-module homomorphism $\psi_2:\Omega_k^1(A)\rightarrow\Omega_k(A)$ such that $\psi_2\delta_{A/k}^1=\delta_{k1}^A$. It now easily follows that $\psi_1\psi_2$ is the identity on $\Omega_k^1(A)$. Thus $\Omega_k^1(A)$ is an A-module direct summand of $\Omega_k(A)$.

Throughout the rest of this paper, k will denote an arbitrary field. Suppose K is a field which contains k. If $\{u_\alpha;\ \alpha\in\Lambda\}$ is a set of indeterminates over K, then we shall denote by $K\langle u_\alpha|\alpha\in\Lambda\rangle$ the commutative ring of polynomials in the u_α, coefficients in K, which have no constant term. Thus if $K[u_\alpha|\alpha\in\Lambda]$ denotes the ordinary ring of polynomials, $K\langle u_\alpha\rangle+K=K[u_\alpha]$. We shall need the following theorem:

THEOREM 1. *Let K be a separable extension of k. Suppose the transcendence degree of K over k is $n\geqq 1$. Let $\{u_{ij}|j=1,\cdots,n,\ i=1,\cdots,\infty\}$ be a collection of indeterminates over K. Then*

$$\Omega_k(K)\cong K\langle u_{ij}|j=1,\cdots,n,\ i=1,\cdots,\infty\rangle.$$

PROOF. Let $x_1,\cdots,x_n$ be elements of K which are algebraically independent over k and such that K is a separable algebraic extension of $k(x_1,\cdots,x_n)$. Now it is known that $\Omega_k(k[x_1,\cdots,x_n])\cong k[x_1,\cdots,x_n]\times\langle u_{ij}\rangle$ [**1**, p. 29]. It follows from [**1**, Theorem 6] that $\Omega_k(k(x_1,\cdots,x_n))\cong k(x_1,\cdots,x_n)\langle u_{ij}\rangle$. Let $S=k(x_1,\cdots,x_n)$. Then the canonical k-higher derivation δ_k^s of S into $\Omega_k(S)$ is given by $\delta_{ki}^s(x_j)=u_{ij}$.

Now we may view δ_k^s as a k-higher derivation of S into $Q(K\langle u_{ij}\rangle)$, the quotient field of $K\langle u_{ij}\rangle$. By using Zorn's lemma and [**3**, Proposition 2], we may uniquely extend δ_k^s to a k-higher derivation $\hat{\delta}$ of K into $Q(K\langle u_{ij}\rangle)$. Since K is a separable algebraic extension of S, one can easily argue that, for all q, $\hat{\delta}_q(K)\subset K\langle u_{ij}\rangle$. Thus $\hat{\delta}$ is a k-higher derivation of K into $K\langle u_{ij}\rangle$.

It now follows easily that $(K\langle u_{ij}\rangle, \hat{\delta})$ has the universal mapping property (c). Hence it follows that $\Omega_k(K) \cong K\langle u_{ij}\rangle$. □

By an affine ring over k, we shall mean any homomorphic image of $k[X_1, \cdots, X_m]$, the polynomial ring in a finite number of indeterminates over k. Let A be a k-algebra. We shall say that $\Omega_k(A)$ is a free A-algebra if $\Omega_k(A)$ is isomorphic as an A-algebra to $A\langle X_\alpha; \alpha \in \Lambda\rangle$. Here the $\{X_\alpha; \alpha \in \Lambda\}$ is a collection of indeterminates indexed by some set Λ and $A\langle X_\alpha; \alpha \in \Lambda\rangle$ denotes the collection of all polynomials in the X_α, coefficients in A, which have no constant term. We note that if $\Omega_k(A)$ is a free A-algebra then $\Omega_k(A)$ is a free A-module. Hence if A is a local ring, the lemma implies that $\Omega_k^1(A)$ is a free A-module if $\Omega_k(A)$ is a free A-algebra.

THEOREM 2. *Let A be an affine ring over k and let R be the quotient ring of A with respect to a prime ideal p. Assume* (1°) *A is an integral domain*; (2°) *the quotient field K of A is a separable extension of k*; (3°) *the residue class field L of R is a separable algebraic extension of k. Then R is a regular local ring if and only if $\Omega_k(R)$ is a free R-algebra.*

PROOF. Let us first suppose that R is a regular local ring. Let $\{z_1, \cdots, z_n\}$ be any regular system of parameters of R. Note that the hypotheses of Theorem 2 imply that p is a maximal ideal of A. Hence $n = \dim R$ is equal to the transcendence degree of K over k. Further it is known that $\{z_1, \cdots, z_n\}$ is a separating transcendence basis of K over k [**4**, Theorem 3, Corollary 2]. Since K is a separable algebraic extension of $k(z_1, \cdots, z_n)$, Theorem 1 implies $\Omega_k(K) = K\langle u_{ij} \mid j=1, \cdots, n, i=1, \cdots, \infty\rangle$ for indeterminates u_{ij} over K. Further we know that $u_{ij} = \delta_{ki}^K(z_j)$. Now $K \otimes_R \Omega_k(R) \cong \Omega_k(K)$ [**1**, Theorem 6] under the K-algebra mapping sending

$$1 \otimes_R \delta_{ki}^R(x) \rightarrow \delta_{ki}^K(x) \quad \text{for } x \text{ in } R.$$

Thus it is clear that $\{\delta_{ki}^R(z_j) \mid j=1, \cdots, n, i=1, \cdots, \infty\}$ is a collection of free elements in $\Omega_k(R)$. That is, a polynomial f in the $\delta_{ki}^R(z_j)$ with coefficients in R is zero if and only if each of the coefficients of f is zero. Hence $\{\delta_{ki}^R(z_j)\}$ forms a free basis for $\Omega_k(R)$ if they generate all of $\Omega_k(R)$ as an R-algebra.

So let $C = R\langle \delta_{ki}^R(z_j); j=1, \cdots, n, i=1, \cdots, \infty\rangle$. Then C is the R-subalgebra of $\Omega_k(R)$ generated by the set $\{\delta_{ki}^R(z_j)\}$. We wish to show that $C = \Omega_k(R)$.

Let $(\Omega_k^1(R), \delta_{R/k}^1)$ denote the universal object for k-derivations of rank one ([**4**] or [**5**]). It follows from the proof of Theorem 3 in [**4**] that $\Omega_k^1(R)$ is generated as an R-module by $\delta_{R/k}^1(z_1), \cdots, \delta_{R/k}^1(z_n)$. Now consider $\delta_{k1}^R: R \rightarrow \Omega_k(R)$. Since δ_{k1}^R is a derivation of R into $\Omega_k(R)$, there exists a unique R-module homomorphism $\psi: \Omega_k^1(R) \rightarrow \Omega_k(R)$ such that $\psi\delta_{R/k}^1 = \delta_{k1}^R$.

Now let x be an element of R. Then there exists $r_1, \cdots, r_n$ in R such that $\delta^1_{R/k}(x)=\sum r_i\delta^1_{R/k}(z_i)$. Thus $\delta^R_{k1}(x)=\psi\delta^1_{R/k}(x)=\sum r_i\psi\delta^1_{R/k}(z_i)=\sum r_i\delta^R_{k1}(z_i)$. Thus the image of δ^R_{k1} is contained in C.

Assume we have shown that the images of $\delta^R_{k1}, \cdots, \delta^R_{km}$ are contained in C. Consider $\delta^R_{k,m+1}: R\to\Omega_k(R)$. Then $\delta^R_{k,m+1}$ induces a derivation δ of rank one from R to the R-module $\Omega_k(R)/C$ in the following way:

$$\delta(x) = \delta^R_{k,m+1}(x) + C, \qquad x \text{ in } R.$$

Clearly δ is a k-linear homomorphism of R into $\Omega_k(R)/C$. Let x and y be elements of R. Then we have

$$\begin{aligned}\delta(xy) &= \delta^R_{k,m+1}(xy) + C = \left(\sum_{i+j=m+1} \delta^R_{ki}(x)\delta^R_{kj}(y)\right) + C \\ &= \left\{x\delta^R_{k,m+1}(y) + y\delta^R_{k,m+1}(x) + \sum_{i+j=m+1;i,j>0} \delta^R_{ki}(x)\delta^R_{kj}(y)\right\} + C.\end{aligned}$$

Now by the induction hypothesis,

$$\sum_{i+j=m+1;i,j>0} \delta^R_{ki}(x)\delta^R_{kj}(y) \in C.$$

Thus

$$\begin{aligned}\delta(xy) &= x\{\delta^R_{k,m+1}(y) + C\} + y\{\delta^R_{k,m+1}(x) + C\} \\ &= x\delta(y) + y\delta(x).\end{aligned}$$

Hence δ is a derivation of R into $\Omega_k(R)/C$. From the universal mapping property of $(\Omega^1_k(R), \delta^1_{R/k})$, there exists a unique R-module homomorphism $\psi_1:\Omega^1_k(R)\to\Omega_k(R)/C$ such that $\delta(x)=\psi_1\delta^1_{R/k}(x)$. Thus the image of δ is generated by $\delta(z_1), \cdots, \delta(z_n)$. This immediately implies that the image of $\delta^R_{k,m+1}$ is contained in C. Thus C contains $\{\delta^R_{ki}(x) \mid x \in R, i\geqq 1\}$. Since $\Omega_k(R)$ is generated as an R-algebra by this set, we get $C=\Omega_k(R)$.

Thus if R is a regular local ring with regular system of parameters $\{z_1, \cdots, z_n\}$ then $\Omega_k(R)$ is a free R-algebra with basis $\{\delta^R_{ki}(z_j)\}$.

The converse follows immediately from the lemma and [**4**, Theorem 3]. □

Since an irreducible subvariety W of an irreducible variety V is simple if and only if the local ring $Q(W/V)$ of W on V is regular, we can rephrase Theorem 2 as follows:

COROLLARY. *Let V be an irreducible affine variety defined over k. Let P be a point of V such that $k(P)$ is a separable algebraic extension of k. Then P is a simple point of V if and only if $\Omega_k(Q(P/V))$ is a free $Q(P/V)$-algebra. Here of course $Q(P/V)$ denotes the local ring of P at V.*

As Y. Nakai has shown in [**4**], if either hypothesis (2°) or (3°) is dropped, $\Omega_k^1(R)$ being a free R-module is no longer equivalent to R being a regular local ring. Hence the hypotheses (2°) and (3°) in Theorem 2 are inevitable.

If R has a separating residue class field L, then we can generalize Theorem 2.

Let R be a local integral domain containing the field k and having quotient field K. Let L be the residue class field of R and assume L is a separable extension of k. Let $\{u_1, \cdots, u_s\}$ be a separating transcendence basis of L/k. Then we can find representatives $\alpha_1, \cdots, \alpha_s$ of $u_1, \cdots, u_s$ in R. The elements $\alpha_1, \cdots, \alpha_s$ are clearly algebraically independent over k and $F=k(\alpha_1, \cdots, \alpha_s)$ is a subfield of R. If it is possible to find u_i's and α_i's as above such that K is a separable extension of F, then we say R has a separating residue class field. If R has a separating residue class field, we shall call $\alpha_1, \cdots, \alpha_s$ separating representatives in R.

THEOREM 3. *Let A be an affine ring over the field k and let R be the quotient ring of A with respect to a prime ideal p in A. Assume* (1°) *A is an integral domain,* (2°) *the quotient field K of R is a separable extension of k,* (3°) *the residue class field L of R is a separable extension of k,* (4°) *R has a separating residue class field. Then R is a regular local ring if and only if $\Omega_k(R)$ is a free R-algebra.*

PROOF. Let us first assume R is a regular local ring. Let $\{z_1, \cdots, z_n\}$ be a regular system of parameters for R and $\{\alpha_1, \cdots, \alpha_s\}$ separating representatives in R. Since the quotient field of A/p is L, we know the transcendence degree of K/k is exactly equal to $n+s$. K is a separable extension of $F=k(\alpha_1, \cdots, \alpha_s)$ and as pointed out in the proof of Theorem 2, $\{z_1, \cdots, z_n\}$ is a separating transcendence basis of K/F. Hence $\{z_1, \cdots, z_n, \alpha_1, \cdots, \alpha_s\}$ is a separating transcendence basis for K/k. Thus it follows from Theorem 1 that $\{\delta_{ki}^K(z_j),\ \delta_{ki}^K(\alpha_l) \mid j=1, \cdots, n, l=1, \cdots, s, i=1, \cdots, \infty\}$ forms a free K-algebra basis for $\Omega_k(K)$. Now $\Omega_k(K) \cong K \otimes_R \Omega_k(R)$. Hence as in Theorem 2, $\{\delta_{ki}^R(z_j), \delta_{ki}^R(\alpha_l)\}$ will form a free R-algebra basis for $\Omega_k(R)$ if they generate all of $\Omega_k(R)$.

Let C be the R-subalgebra of $\Omega_k(R)$ generated by $\{\delta_{ki}^R(z_j), \delta_{ki}^R(\alpha_l)\}$. We shall first show that the image of $\delta_{k1}^R: R \to \Omega_k(R)$ is continued in C. As before, let $\Omega_k^1(R)$ denote the universal object for k-derivations of rank one on R. Then it follows from [**4**, Proposition 3] that

$$*: 0 \longrightarrow R \otimes_F \Omega_k^1(F) \xrightarrow{\mu} \Omega_k^1(R) \xrightarrow{\alpha} \Omega_F^1(R) \longrightarrow 0$$

is an exact sequence of R-modules. Here μ is the R-module homomorphism given by $\mu(1 \otimes \delta_{F/k}^1(x)) = \delta_{R/k}^1(x)$ and α can be taken to be the R-module homomorphism which sends $\delta_{R/k}^1(x)$ to $\delta_{R/F}^1(x)$.

Now by [**4**, Theorem 3], $\Omega^1_F(R)$ is a free R-module generated by $\delta^1_{R/F}(z_1), \cdots, \delta^1_{R/F}(z_n)$. $\Omega^1_k(F)$ is a free F-module generated by $\delta^1_{F/k}(\alpha_1), \cdots, \delta^1_{F/k}(\alpha_s)$. Hence it follows that $\Omega^1_k(R)$ is a free R-module with basis $\delta^1_{R/k}(z_1), \cdots, \delta^1_{R/k}(z_n), \delta^1_{R/k}(\alpha_1), \cdots, \delta^1_{R/k}(\alpha_s)$.

From the universal mapping property of $(\Omega^1_k(R), \delta^1_{R/k})$ we get a unique R-module homomorphism $\psi:\Omega^1_k(R)\to\Omega_k(R)$ such that for all x in R, $\delta^R_{k1}(x)=\psi\delta^1_{R/k}(x)$. As in the proof of Theorem 2, we get that the image of δ^R_{k1} is contained in $\sum R\delta^R_{k1}(z_j)+\sum R\delta^R_{k1}(\alpha_l)$ which in turn is contained in C. The proof now follows as in Theorem 2 and we get $C=\Omega_k(R)$. Thus if R is a regular local ring, and $\{z_1, \cdots, z_n\}$ is any regular system of parameters of R, then $\{\delta^R_{ki}(z_j), \delta^R_{ki}(\alpha_l)\}$ forms a free R-algebra basis for $\Omega_k(R)$.

As in Theorem 2, the converse follows from the lemma and [**4**, Theorem 3′]. □

COROLLARY. *Let V be an irreducible affine variety defined over the field k. Let P be a point of V (not necessarily algebraic over k). Assume $Q(P/V)$ has a separating residue class field. Then P is a simple point of V if and only if $\Omega_k(Q(P/V))$ is a free $Q(P/V)$-algebra.*

REFERENCES

1. W. C. Brown, *The algebra of differentials of infinite rank* (to appear).

2. W. C. Brown and W. E. Kuan, *Ideals and higher derivations in commutative rings*, Canad. J. Math. (to appear).

3. N. Heerema, *Higher derivations and automorphisms of complete local rings*, Bull. Amer. Math. Soc. **76** (1970), 1212–1225, MR **42** #1818.

4. Y. Nakai, *On the theory of differentials in commutative rings*, J. Math. Soc. Japan **13** (1961), 63–84. MR **23** #A2437.

5. ———, *Higher order derivations*. I, Osaka J. Math. **7** (1970), 1–27. MR **41** #8404.

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823