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# EUCLIDEAN NEIGHBORHOODS IN COMPACT SEMIGROUPS

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ABSTRACT. In this paper sufficient conditions are given under which a maximal idempotent in a compact connected semigroup cannot have a two-dimensional Euclidean neighborhood.

1. **Introduction.** It has been shown by Mostert and Shields [7] that the identity element of a compact connected semigroup cannot have a Euclidean neighborhood unless $S$ is a group. Later Cohen and Koch [1] proved that a right identity in $S$ cannot have a Euclidean neighborhood unless $S$ is left simple. These results are now corollaries to the fact that a right identity in a compact connected semigroup with a zero is peripheral [2, p. 168]. In this paper we use the concept of peripherality to obtain sufficient conditions under which maximal idempotents fail to have two-dimensional Euclidean neighborhoods. For an expository discussion of peripherality and its applications in this context, the reader is referred to [4].

2. **Preliminary remarks.** The notation of this paper is that of [8]. In particular $S$ will denote a topological semigroup, $K$ its minimal ideal, and $E$ is used to denote the set of idempotents. All spaces are assumed to be compact Hausdorff. The cohomology used is Alexander-Wallace-Spanier theory with coefficient group arbitrary and reduced groups in dimension zero. If $A$ is a closed subset of a space $X$ and $h \in H^n(X)$, then $h|A$ denotes the image of $h$ under the natural homomorphism $H^n(X) \to H^n(A)$. We recall that a floor for an element $h$ of $H^n(X)$ is a closed subspace $F$ of $X$ such that $h|F \neq 0$ but $h|A = 0$ for each proper closed subset $A$ of $F$. If $h|B \neq 0$ for some closed set $B$ in $X$, then a floor for $h$ exists and may be chosen as a subset of $B$. The following lemma gives sufficient conditions that a floor for a nonzero element $h$ of $H^n(X)$ be unique.

2.1 LEMMA. *Let $X$ be a space, $A$ and $B$ closed subsets of $X$ such that $X = A \cup B$ and $A \cap B = \{p\}$. If $h_0 \in H^n(A)$ and $A$ is a floor for $h_0$, then there*

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*exists an element $h$ in $H^n(X)$ such that* (i) $h|A=h_0$, *and* (ii) *if $C$ is a closed subset of $X$ such that $h|C\neq 0$, then $A\subset C$.*

PROOF. Consider the Mayer-Vietoris sequence

$$\longrightarrow H^{n-1}(A\cap B)\longrightarrow H^n(X)\xrightarrow{J^*} H^n(A)\times H^n(B)\longrightarrow H^n(A\cap B)\longrightarrow.$$

$J^*$ is onto so there exists $h\in H^n(X)$ such that $J^*(h)=(h_0, 0)$; i.e. $h|A=h_0$ and $h|B=0$. Suppose $C$ is a closed subset of $X$ such that $h|C\neq 0$ and consider the exact sequence

$$\longrightarrow H^{n-1}(A\cap B\cap C)\longrightarrow H^n(C)\xrightarrow{J^*} H^n(A\cap C)\times H^n(B\cap C)\longrightarrow.$$

Since $J^*$ is 1-1, $J^*(h|C)\neq 0$. Since $h|B\cap C=0$, we have $h|A\cap C\neq 0$. But $A$ is a floor for $h_0$ and $h|A\cap C=h_0|A\cap C$, so $A\cap C=A$. This establishes the conclusion of the theorem.

A point $p$ in a space $X$ is peripheral if there exist small open sets $V$ containing $p$ such that the induced mapping $H^n(X)\to H^n(X\backslash V)$ is an isomorphism for all $n\geqq 0$. That is to say there exist small open sets containing $p$ such that $H^n(X, X\backslash V)=0$ for all $n\geqq 0$. By the map excision theorem it follows that $p$ is peripheral if and only if $H^n(V^*, V^*\backslash V)=0$ for all $n\geqq 0$. The local nature of the concept of peripherality is thus clearly seen; i.e. if $A$ is a closed subset of a space $X$ and $p\in A^\circ$, then $p$ is peripheral in $A$ if and only if $p$ is peripheral in $X$. These remarks together with the aforementioned result concerning a right identity in a semigroup yield the following lemma.

2.2 LEMMA. *Let $S$ be a compact connected semigroup with a zero and suppose $e$ and $f$ are idempotents in $S$ such that $S=Se\cup Sf$. Then either $e\in Sf$, in which case $S=Sf$, or $e$ is peripheral in $S$.*

The following theorem is proved in [5] and is of importance to the sequel.

2.3 THEOREM. *If $S$ is a compact connected semigroup with a zero satisfying $S=ESE$, and $h$ is a nonzero element of $H^2(S)$, then there exists a pair of idempotents $e$ and $f$ such that $h|Se\cup Sf\neq 0$.*

Recall that the $\leqq(\mathscr{L})$ & $\leqq(\mathscr{J})$ orderings in a semigroup $S$ are defined as follows: for $x, y\in S$, (i) $x\leqq(\mathscr{L})y$ if $x\cup Sx\subset y\cup Sy$, and (ii) $x\leqq(\mathscr{J})y$ if $x\cup Sx\cup xS\cup SxS\subset y\cup Sy\cup yS\cup SyS$. We say that an element $x$ in $S$ is $\mathscr{L}$-maximal ($\mathscr{J}$-maximal) if $x$ is maximal relative to the ordering $\leqq\mathscr{L}$ ($\leqq\mathscr{J}$). It is known that a $\mathscr{J}$-maximal idempotent is $\mathscr{L}$-maximal in a compact semigroup. This together with the preceding remarks yield the following corollary to Lemma 2.2.

2.4 COROLLARY. *If $S$ is a continuum with a zero, $S=ESE$, and $e$ is a $\mathcal{J}$-maximal idempotent of $S$, then $e$ does not belong to the unique floor of any nonzero element of $H^2(S)$.*

PROOF. Suppose that $h \in H^2(S)$ and $F$ is the unique floor for $h$. First one should observe that no point of $F$ is peripheral in any closed subset $A$ of $S$ which contains $F$. Indeed if $p \in F$ and $V$ is any $A$-open set containing $p$, then $h|A\backslash V=0$ since $F$ is the unique floor for $h$. Therefore the natural homomorphism $H^n(A) \to H^n(A\backslash V)$ is not 1-1 and so $p$ is not peripheral in $A$.

Now by Theorem 2.3 there exist idempotents $g$ and $f$ in $S$ such that $h|Sg \cup Sf \neq 0$, and therefore $F \subset Sg \cup Sf$. Thus, if $e$ is a $\mathcal{J}$-maximal idemptoent and $e \in F$, then it may be assumed that $e \in Sg$. Since $e$ is also $\mathcal{L}$-maximal it follows that $Se=Sg$, so $F \subset Se \cup Sf$. Because $h|Sf=0$ and $h|Se \cup Sf \neq 0$ it must be that $e \notin Sf$, so by Lemma 2.2, $e$ is peripheral in $Se \cup Sf$. This contradicts the above and the proof is complete.

The following lemma concerns the topology of the plane and a proof is only sketched here.

2.5 LEMMA. *Let $\boldsymbol{R}^2$ denote the Euclidean plane. Suppose $N$ is a closed totally disconnected set in $\boldsymbol{R}^2$ containing the distinguished point $p$. Then there exists a compact neighborhood $M$ of $p$ satisfying*: (i) $F(M) \cap N = \square$, (ii) *$F(M)$ is connected*, (iii) *if $C$ is any closed connected subset of $M$ containing $F(M)$ and $D$ is any component of $M\backslash C$, then $D$ is homeomorphic to the open unit disk.*

PROOF. Let $\Omega$ be the point at infinity. Applying Theorem 20 of R. L. Moore's Chapter IV to the points $p$ and $\Omega$ of the closed and totally disconnected subset $N \cup \{\Omega\}$ of the sphere $\boldsymbol{R}^2 \cup \{\Omega\}$, one obtains a simple closed curve $J$ such that $p$ is in the interior of $J$ and $J \cap (N \cup \{\Omega\}) = \varnothing$. Letting $M=J$ and its interior, claims (i), (ii) and (iii) follow immediately.

3. **Principal results.** We proceed to give sufficient conditions under which maximal idempotents in compact semigroups fail to have two-dimensional Euclidean neighborhoods. The motivation is to give conditions under which we may find an ideal $J$ in $S$ such that in the Rees quotient $S/J$, $e$ lies in a 2-sphere neighborhood; thus establishing a contradiction to Corollary 2.4. We give the main theorem now.

3.1 THEOREM. *Let $S$ be a continuum satisfying $S=ESE$. If $e$ is a $\mathcal{J}$-maximal idempotent of $S$ and $\mathcal{J}e$ is totally disconnected, then $e$ does not lie in a two-dimensional Euclidean neighborhood.*

PROOF. Suppose $e$ is an element of a two-dimensional neighborhood $V$. Since $\mathcal{J}e$ is totally disconnected, $e \notin K$. Let $M$ be a neighborhood of

$e$ satisfying the conclusion of Lemma 2.5. (Here $N$ is taken to be $\mathcal{J}_e \cap V$ and $e$ the distinguished point.) It is well known that $S \backslash \mathcal{J}_e$ is a maximal ideal of $S$, and since $F(M) \cap \mathcal{J}_e = \square$, the ideal $SF(M)S$ is contained in $S \backslash \mathcal{J}_e$. Moreover $F(M) \subset SF(M)S$ since $S = ESE$, and because $F(M)$ is connected, the set $C = SF(M)S \cap M$ is connected. Letting $D$ denote the component of $M \backslash C$ containing $e$, we have that $D$ is homeomorphic to the open unit disk of $\boldsymbol{R}^2$.

Let $J = SF(M)S$, and consider the Rees quotient modulo $J$ with natural map $p: S \to S/J$. Clearly $p(D)$ is homeomorphic to $D$ and it is easily verified that $p(D)$ is open in $S/J$, and $p(D)^* \backslash p(D) = p(J)$; i.e. $p(D)$ is homeomorphic to an open disk and its boundary is a point. Therefore $p(D)^*$ is homeomorphic to a 2-sphere. The hypotheses of Lemma 2.1 are satisfied so $p(D)^*$ is the unique floor for some element $h$ in $H^2(p(S))$. The fact that $p(e)$ is a maximal idempotent in $S/J$, together with the preceding statement, yields a contradiction to Corollary 2.4. Therefore $e$ does not have a two-dimensional Euclidean neighborhood.

There are several instances where a maximal idempotent $e$ fails to have a two-dimensional Euclidean neighborhood. In particular, let $S$ be a continuum satisfying $S^2 = S$ and $S \neq K$. If a maximal idempotent $e$ has a two-dimensional Euclidean neighborhood and $H_e$ is totally disconnected then so is $\mathcal{J}_e$. If in addition to the above, $S = SeS$, then $H_e$ is necessarily totally disconnected [6]. We thus obtain the following corollaries:

3.2 Corollary. *Let $S$ be a continuum with $S = ESE$, and suppose $e$ is a maximal idempotent in $S$. If $\mathcal{H}_e$ is totally disconnected and $e$ has a two-dimensional Euclidean neighborhood, then $S = K$.*

3.3 Corollary. *Let $S$ be a continuum with $S = ESE$ and $S = SeS$ for some (maximal) idempotent $e$ in $S$. If $e$ has a two-dimensional Euclidean neighborhood, then $S = K$.*

3.4 Corollary. *Let $S$ be a continuum satisfying $S = ESE$, and suppose $e$ is a $\mathcal{J}$-maximal idempotent of $S$. If $S$ is a subset of the plane $\boldsymbol{R}^2$ and $S \neq K$, then $e$ lies on the boundary of $S$ in $\boldsymbol{R}^2$.*

Proof. If $\mathcal{H}_e$ is totally disconnected, the conclusion follows from Corollary 3.2. Otherwise $e$ lies on a circle subgroup $G_0$ of $H_e$ which separates $\boldsymbol{R}^2$. But $G_0$ does not separate $S$ [3], so the conclusion follows.

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