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SPARSE SUBSETS OF ORTHONORMAL SYSTEMS

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ABSTRACT. There exist families of Walsh, Haar and trigonometric functions that have density zero and yet are complete in the sense of measure.

1. **Introduction.** Goffman and Waterman proved elegantly [**1**] that from a system of functions which is total in measure on [0, 1], one may discard a suitable infinite subset without destroying totality. It is natural to ask "how many" functions may be discarded from specific systems or equivalently, how "thin" a subset can be and still be total in measure. In this regard, we have proved [**2**] that a system $\{\psi_{n_i}\}$ of Walsh functions is total in measure on [0, 1] if it has density 1, where

$$\text{density} = \limsup_k \limsup_n \frac{\Lambda(n+k) - \Lambda(n)}{k},$$

$\Lambda(n)$ denoting the number of indices n_i for which $n_i \leqq n$.

In this paper we show there are actually Walsh families of density zero that are total in measure on [0, 1], and that the same is true for the trigonometric and Haar functions:

THEOREM. *There exist subsets of density zero of the Walsh functions, the trigonometric functions* $\{\cos 2\pi nx, \sin 2\pi nx\}$, *and the Haar functions that are total in measure on* [0, 1].

We shall give separate proofs for the three systems in the next three sections.

NOTATION. If Φ is a set of functions, $L(\Phi)$ will denote the set of all finite real linear combinations of elements of Φ. The word "total" will always mean "total in measure".

2. **Walsh functions.** We use a well-known property of the Dirichlet kernels of the Walsh system:

$$D_n(x) = 1 + \sum_{j=1}^{2^n-1} \psi_j(x) = 2^n, \qquad x \in [0, 2^{-n}),$$
$$= 0, \qquad x \in [2^{-n}, 1).$$

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If $g_n(x)=1-D_n(x)$, then

$$g_n(x) = -\sum_{j=1}^{2^n-1} \psi_j(x) = 1 - 2^n, \qquad x \in [0, 2^{-n}),$$
$$= 1, \qquad x \in [2^{-n}, 1).$$

Clearly $g_n(x)\to 1$ in measure on [0, 1].

We define

$$f_n(x) = g_n(2^{n^2}x) = \sum_{j=1}^{2^n-1} \psi_j(2^{n^2}x).$$

Then $f_n(x)\to 1$ in measure on [0, 1]. Furthermore, since for any Walsh function $\psi_j(x)$, $\psi_j(2^k x)=\psi_{j2^k}(x)$, we have $f_n \in L(\Psi_n)$ where

$$\Psi_n = [\psi_k : k = j \cdot 2^{n^2}, j = 1, 2, \cdots, 2^n - 1].$$

Now let $\varphi_1, \varphi_2, \varphi_3, \cdots$ denote the sequence of Walsh functions $\psi_0, \psi_1, \psi_0, \psi_1, \psi_2, \cdots$ in which each Walsh function occurs infinitely often. Define $\Psi = \bigcup_{n=1}^{\infty} \varphi_n \Psi_n$. The set $\varphi_n\Psi_n$ consists of Walsh functions whose indices are in arithmetic progression, with common difference 2^{n^2}. It follows that Ψ has density zero. To see that Ψ is total, note that for each Walsh function ψ_k, there is a sequence $\{n_i\}$, depending on k, for which $\psi_k \Psi_{n_i} \subset \Psi$. Therefore,

$$\psi_k f_{n_i} \in L(\psi_k \Psi_{n_i}) \subset L(\Psi).$$

But $f_{n_i}(x)\to 1$, so $\psi_k(x) f_{n_i}(x) \to \psi_k(x)$ in measure on [0, 1]. Thus each Walsh function can be approximated arbitrarily closely in measure by elements of $L(\Psi)$. Therefore Ψ is total.

3. **Trigonometric functions.** As in the preceding section, we construct a sequence $\{f_n(x)\}$ such that $f_n(x)\to 1$ in measure on [0, 1]. This time we use the fact that $K_n(x)\to 0$ in measure on [0, 1], where $K_n(x)$ is the Fejer kernel.

Let $\{m_n\}$ be a sequence of positive integers increasing so rapidly that $m_{n+1} > m_n^2$, for instance $m_n = 2^{n!}$. Define

$$f_n(x) = 1 - K_{m_n}(m_n x) = \sum_{k=1}^{m_n} a_{kn} \cos(2\pi m_n kx).$$

Then $f_n(x)\to 1$ in measure on [0, 1] and $f_n \in L(\Phi_n)$ where

$$\Phi_n = [\cos 2\pi jx : j = m_n, 2m_n, \cdots, m_n^2].$$

Let $\{k_n\}$ denote the sequence 0, 1, 0, 1, 2, 0, 1, 2, 3, $\cdots$. Define

$$\Phi_n' = [\sin 2\pi jx, \cos 2\pi jx : j = m_n \pm k_n, 2m_n \pm k_n, \cdots, m_n^2 \pm k_n]$$

and $\Phi = \bigcup_{n=1}^{\infty} \Phi_n'$. It is clear that Φ has density zero.

For each positive integer j, there is a sequence $\{n_i\}$ depending on j, such that

$$(\sin 2\pi jx)\Phi_{n_i} \in L(\Phi'_{n_i}), \qquad (\cos 2\pi jx)\Phi_{n_i} \in L(\Phi'_{n_i}).$$

It follows that

$$(\sin 2\pi jx)f_{n_i}(x), (\cos 2\pi jx)f_{n_i}(x) \in L(\Phi'_{n_i}) \subset L(\Phi).$$

Since $f_{n_i}(x) \to 1$ in measure on $[0, 1]$,

$$(\sin 2\pi jx)f_{n_i}(x) \to \sin 2\pi jx, \qquad (\cos 2\pi jx)f_{n_i}(x) \to \cos 2\pi jx.$$

Hence each function $\sin 2\pi jx$, $\cos 2\pi jx$ can be approximated arbitrarily closely in measure by elements of $L(\Phi)$. Therefore Φ is total.

4. **Haar functions.** Here we use a criterion proved by Robert E. Zink and the author [**3**]: Suppose $\{h_j\}$ is a family of Haar functions. Let $\sigma(h_j)$ denote the support of h_j and let $E = \limsup \sigma(h_j)$. Then $\{h_j\}$ is total on a set $G \subset [0, 1]$ if and only if $|G| = |G \cap E|$. In view of this criterion we shall exhibit a subset H of the Haar functions with density zero and such that $|E| = 1$.

Let B_n denote the block of Haar functions $[h_j : 2^n \leqq j < 2^{n+1}]$. For each positive integer k, let G_k be the union of any collection of 2^k blocks B_n, subject to two conditions:

(a) If B_n, $B_m \in G_k$, then $|n-m| > k$.

(b) If $B_n \in G_k$ and $B_m \in G_{k+1}$, then $n < m$.

(These conditions imply $n > 2^k$ when $B_n \in G_k$ and $k > 2$.)

Now we define the desired system H of Haar functions. For each k let $H_k = [h_j : h_j \in G_k$ and $n \equiv 0 \pmod{2^k}]$; then set $H = \bigcup_{k=1}^{\infty} H_k$. Clearly H has density zero (just as Ψ has density zero in §2).

For each B_n that occurs in some G_k, let

$$E_n = \bigcup \sigma(h_j) \qquad (h_j \in B_n \cap H_k).$$

It is not hard to see that

$$E_n = [x : d_i(x) = 0, n - k + 1 \leqq i \leqq n]$$

where $d_i(x)$ is the ith dyadic digit of x. Thus $|E_n| = 2^{-k}$. Because of (a) and (b), each E_n is defined by conditions on different sets of dyadic digits, hence the E_n are independent sets. Since there are 2^k sets E_n for each k,

$$\sum |E_n| = \sum_k \sum_{B_n \in G_k} |E_n| = \sum_k (2^k \cdot 2^{-k}) = \sum 1 = \infty.$$

It follows by the Borel-Cantelli Lemma that $|\limsup E_n| = 1$ which implies that H is total on $[0, 1]$.

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